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Factor-set of binary matrices and Fibonacci numbers

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ABSTRACT

The article discusses the set of square $n \times n$ binary matrices with the same number of 1's in each row and each column. An equivalence relation on this set is introduced. Each binary matrix is represented using ordered n-tuples of natural numbers. We are looking for a formula which calculates the number of elements of each factor-set by the introduced equivalence relation. We show a relationship between some particular values of the parameters and the Fibonacci sequence.

1. Introduction

A *binary* (or *boolean*, or (0,1)-*matrix*) is a matrix whose all elements belong to the set $\mathcal{B} = \{0, 1\}$. With $\mathcal{B}_n$ we will denote the set of all $n \times n$ binary matrices.

Let n and k be integers such that $0 \leqslant k \leqslant n, n \geqslant 2$. We let Λ_n^k denote the set of all $n \times n$ binary matrices in each row and each column of which there are exactly k in number 1's. Let us denote with $\lambda(n,k) = |\Lambda_n^k|$ the number of all elements of Λ_n^k.

There is not any known formula to calculate the $\lambda(n,k)$ for all n and k. There are formulas for the calculation of the function $\lambda(n,k)$ for each n for relatively small values of k, more specifically, for $k = 1, k = 2$ and $k = 3$. We do not know any formula to calculate the function $\lambda(n,k)$ for $k > 3$ and for all positive integer n.

It is easy to prove the following well-known formula

$$\lambda(n,1) = n!.$$

The following formula

$$\lambda(n,2) = \sum_{2x_2+3x_3+\cdots+nx_n=n} \frac{(n!)^2}{\prod_{r=2}^{n} x_r!(2r)^{x_r}}$$

is well known [8].

One of the first recursive formulas for the calculation of $\lambda(n,2)$ appeared in [1] (see also [4, p. 763]).

$$\left| \begin{array}{l} \lambda(n,2) = \frac{1}{2} n(n-1)^2 \left[(2n-3)\lambda(n-2,2) + (n-2)^2 \lambda(n-3,2) \right] \quad \text{for } n \geqslant 4, \\ \lambda(1,2) = 0, \quad \lambda(2,2) = 1, \quad \lambda(3,2) = 6. \end{array} \right.$$

Another recursive formula for the calculation of $\lambda(n,2)$ occurs in [3].

$$\left| \begin{array}{l} \lambda(n,2) = (n-1)n\lambda(n-1,2) + \frac{(n-1)^2 n}{2} \lambda(n-2,2) \quad \text{for} \quad n \geqslant 3, \\ \lambda(1,2) = 0, \quad \lambda(2,2) = 1. \end{array} \right.$$

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The next recursive system is to calculate $\lambda(n,2)$.

$$\begin{vmatrix} \lambda(n+1,2) = n(2n-1)\lambda(n,2) + n^2\lambda(n-1,2) - \pi(n+1);\ n \geqslant 2, \\ \pi(n+1) = \frac{n^2(n-1)^2}{4}[8(n-2)(n-3)\lambda(n-2,2) + (n-2)^2\lambda(n-3,2) - 4\pi(n-1)];\ n \geqslant 4, \\ \lambda(1,2) = 0, \quad \lambda(2,2) = 1, \quad \pi(1) = \pi(2) = \pi(3) = 0, \quad \pi(4) = 9, \end{vmatrix}$$

where $\pi(n)$ identifies the number of a special class of Λ_n^2-matrices [9].

The following formula is an explicit form for the calculation of $\lambda(n,3)$.

$$\lambda(n,3) = \frac{n!^2}{6^n}\sum \frac{(-1)^\beta(\beta+3\gamma)!2^\alpha 3^\beta}{\alpha!\beta!\gamma!^2 6^\gamma},$$

where the sum is done as regards all $\frac{(n+2)(n+1)}{2}$ solutions in nonnegative integers of the equation $\alpha+\beta+\gamma = n$ [7]. As it is noted in [6], this formula does not give us good opportunities to study behavior of $\lambda(n,3)$.

Let $A, B \in \Lambda_n^k$. We will say that $A \sim B$, if A is obtained from B by moving some rows and/or columns. Obviously, the relation defined like that is an equivalence relation. We denote with

$$\mu(n,k) = \left|\Lambda^k_{n/\sim}\right|, \tag{1}$$

the number of equivalence classes on the above defined relation.

Problem 1. Find $\mu(n,k)$ for given integers n and $k, 1 \leqslant k < n$.

The task of finding the cardinal number $\mu(n,k)$ of the factor set $\left|\Lambda^k_{n/\sim}\right|$ for all integers n and $k, 1 \leqslant k \leqslant n$ is an open scientific problem. This is the subject of discussion in this article. We will prove that there is a relationship between some particular values of the parameters of $\mu(n,k)$ of and the Fibonacci sequence. Specifically, we will prove that if $k = n-2$ then $\mu(n,k)$ coincides with the Fibonacci numbers for all $n \in \mathbb{N}, n \geqslant 2$.

2. Canonical binary matrices

Let $\mathbb{N}$ be the set of natural numbers and let

$$\mathcal{T}_n = \left\{\langle a_1, a_2, \ldots, a_n\rangle \mid a_i \in \mathbb{N},\ 0 \leqslant a_i \leqslant 2^n - 1,\ i = 1, 2, \ldots, n\right\}. \tag{2}$$

With "$<$" we will be denoting the lexicographic orders in $\mathcal{T}_n$.

Let us consider the one-to-one correspondence

$$\varphi : \mathcal{B}_n \cong \mathcal{T}_n, \tag{3}$$

which is based on the binary presentation of the natural numbers. If $A \in \mathcal{B}_n$ and $\varphi(A) = \langle a_1, a_2, \ldots a_n\rangle$, then ith row of A is integer a_i written in binary notation.

Definition 1. Let $A \in \mathcal{B}_n$. With $r(A)$ we will be denoting the ordered n-tuple $\langle x_1, x_2, \ldots, x_n\rangle$, where $x_i \in \mathbb{N}, 0 \leqslant x_i \leqslant 2^n - 1, i = 1, 2, \ldots n$ and x_i is the natural number written in a binary notation with the help of the ith row of A. Likewise with $c(A)$ we are denoting the ordered n-tuple $\langle y_1, y_2, \ldots, y_n\rangle$, where $0 \leqslant y_j \leqslant 2^n - 1, j = 1, 2, \ldots n$, and y_j is a natural number written in binary notation with the help of the jth column of A.

Lemma 1. *Let $A \in \mathcal{B}_n, r(A) = \langle x_1, x_2, \ldots, x_n\rangle, c(A) = \langle y_1, y_2, \ldots, y_n\rangle$ and let $1 \leqslant u < v \leqslant n$.*

(a) *Let $x_u > x_v$ and let A' be obtained from A by changing the locations of rows with the number u and v, while the remaining rows stay in their places. Then $c(A') < c(A)$.*
(b) *Let $y_u > y_v$ and let A' be obtained from A by changing the locations of columns with the number u and v, while the remaining columns stay in their places. Then $r(A') < r(A)$.*

Proof.

(a) It is easy to see that $r(A') < r(A)$. Let $A = [a_{ij}]_{n\times n}$, where $a_{ij} \in \{0,1\}, 1 \leqslant i,j \leqslant n$. Then the representation of x_u and x_v in binary notation be respectively as follows:

$$x_u = a_{u1}a_{u2}\cdots a_{un},$$

$$x_v = a_{v1}a_{v2}\cdots a_{vn}.$$