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Summation of power series with rational coefficients via divided differences

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ABSTRACT

We obtain new identities involving divided differences and, as an application, we provide a closed form representation of power series with rational coefficients in terms of divided differences involving special functions.

1. Introduction and related results

There is a lot of mathematical work on power series with rational coefficients describing their properties and the unexpected connection with various branches of science (see e.g., [4,2,10,8,9,5,16,14]).

In [3], the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+n)}$$

is generalized to

$$\sum_{k=0}^{\infty} \frac{1}{(a+kb)\cdots(a+kb+n)} = \frac{1}{n!}\int_0^1 \frac{t^{a-1}(1-t)^n}{1-t^b}\,\mathrm{d}t, \tag{1}$$

where a and b are positive real numbers and n is a positive integer. This problem and generalizations have appeared in various places in the literature (see e.g., Knopp's book [13, p. 234])

$$\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+q)\cdots(a+k+nq)} = \frac{1}{nq}\sum_{i=0}^{q-1} \frac{1}{(a+i)(a+i+q)\cdots(a+i+(n-1)q)}$$

where $n,\ q \geq 1$ are integers and $a > 0$. See also [11, 0.243, 2.[7]].

In [6,7], Efthimiou offers a comprehensive study and solution methods that allows one to find exact values for a large class of convergent series of rational terms. See also [15].

Throughout this paper, let $n \geq 1$ be an integer and let $\mathbb{D} = \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ denote the set of all complex numbers except negative integers. We consider a rational function R of the form

$$R(k) = \frac{P(k)}{(k+t_0)(k+t_1)\cdots(k+t_n)}, \quad k = 0, 1, \ldots$$

where $t_0, t_1, \ldots, t_n \in \mathbb{D}$, are *not necessarily distinct*, and, without loss of generality, P is a polynomial of degree at most $n-1$.

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Our aim is to provide a closed form representation of the sum of the power series

$$\sum_{k=0}^{\infty} R(k) z^k, \quad |z| \leqslant 1,$$

in terms of divided differences involving the Lerch and Polygamma functions. As a byproduct of our result, we obtain a series representation of the divided difference of the Digamma function and an unexpected identity involving the finite difference of the Digamma function and the Euler Beta function.

We will prove, for example, that

$$\sum_{k=0}^{\infty} \frac{1}{(k+t_0)(k+t_1)\cdots(k+t_n)} = (-1)^{n+1}[t_0, t_1, \ldots, t_n; F],$$

where F is the Digamma function.

For the benefit of the reader not familiar with the field, we present a brief background on divided differences, and on Lerch and Polygamma functions.

1.1. The divided differences

Let $t_0, \ldots, t_n$ be distinct complex numbers, $n \geq 1$, and let $f : \{t_0, \ldots, t_n\} \to \mathbb{C}$. We denote by $L[t_0, \ldots, t_n; f]$ and by $[t_0, \ldots, t_n; f]$ the Lagrange interpolating polynomial, respectively the divided difference associated to f on the knots $t_0, \ldots, t_n$. There are several ways to introduce the notion of divided difference, including the one suggested by their very name.

In most books on Numerical Analysis divided differences for distinct points are defined recursively:

$$[t_0; f] = f(t_0), \ldots, [t_0, \ldots, t_n; f] = \frac{[t_1, \ldots, t_n; f] - [t_0, \ldots, t_{n-1}; f]}{t_n - t_0}. \tag{2}$$

In the case of coalescing points, if f is a suitably differentiable function, the divided difference can be defined by using (2) and a limiting process, e.g.,

$$[t_0, t_0; f] = \lim_{t_1 \to t_0} [t_0, t_1; f] = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0} = f'(t_0).$$

For notational purpose, we denote sometimes the divided difference $[t_0, \ldots, t_n; f]$ by $[t_0, \ldots, t_n; f(t)]_t$.

Among the hundreds of formulas involving divided differences we recall the Popoviciu-Steffensen Leibniz-type formula,

$$[t_0, \ldots, t_n; fg] = \sum_{k=0}^{n} [t_0, \ldots, t_k; f] \cdot [t_k, \ldots, t_n; g], \tag{3}$$

and the integral representation

$$[t_0, \ldots, t_n; f] = \frac{1}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z-t_0)\ldots(z-t_n)}, \tag{4}$$

(where the points $t_0, \ldots, t_n$ are inside the simple closed curve C and f is analytic on, and inside C) to mention just two.

1.2. The Digamma and Lerch functions

The (Psi) Digamma F function is defined as the logarithmic derivative of the Euler Gamma function Γ (see e.g., [1, 6.3.1]):

$$\psi(x) = F(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

The Lerch zeta-function, sometimes called the Hurwitz–Lerch zeta-function, is a special function defined by

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}, \quad |z| < 1,\ s \geq 1,\ a > 0.$$

The trigamma function ψ_1 is the derivative of the Digamma function.

2. Auxiliary results

From [11, 0.244(1)] and [11, 3.231(5)], we deduce

$$\sum_{k=0}^{\infty} \frac{1}{(a+k)(b+k)} = \frac{F(b) - F(a)}{b-a} = \frac{-1}{b-a} \int_0^1 \frac{t^{b-1} - t^{a-1}}{1-t} \mathrm{d}t, \quad a, b > 0,\ a \neq b. \tag{5}$$

(see also [6, (7)]).