Applied Mathematics and Computation 238 (2014) 50–56

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability of the zero solution of a family of functional-differential equations

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ARTICLE INFO

Keywords:
Functional-differential equation
Bounded solution
Stability

ABSTRACT

We give some sufficient conditions under which the zero solution of the next functional-differential equation

$$x'(t) = ax(t) + bx(\tau_0(t)) + \sum_{j=1}^{k} c_j x'(\tau_j(t)) + f(x(t), x(\tau_{k+1}(t)), \ldots, x(\tau_{k+l}(t))),$$

where $a, b, c_j,\ j = \overline{1,k}$ are real numbers, $f : \mathbb{R}^{l+1} \to \mathbb{R}$ is a continuous function such that $f(0,\ldots,0) = 0$, and $\tau_j(t) \in C^2[0,\infty),\ j = \overline{0,k+l}$, is asymptotically stable.

1. Introduction

Studying functional-differential equations or systems is a topic of a great interest (see, for example, [1–13,15–22,27–35] and the related references therein). Among them equations and systems which are not solved or are partially solved with respect to the highest derivatives of dependent variables is an area of a considerable interest (see, for example, [2,3,5,6,8–13,16–22,27–32,34,35] and the related references therein).

Special cases of the next functional-difference equation

$$x'(t) = ax(t) + bx(\tau(t)) + cx'(\tau(t)) + f(x(t), x(\tau(t))), \tag{1}$$

where t belongs to an interval $I \subseteq \mathbb{R}$, a, b, c are real numbers, $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $\tau(t)$ is a deviation function, have attracted some attention (see, for example, [2,3,7,8,13,15,18]).

Asymptotic behavior of solutions of equation (1) when $\tau(t) = dt, c = 0$, and $f \equiv 0$ was studied in [15]. Case $\tau(t) = dt, a = 0, c = 0$, and $f \equiv 0$, was investigated in [7]. The existence of analytic almost periodic solutions of the equation when $\tau(t) = dt, c = 0$, and $f \equiv 0$ was considered in [13]. Case $|c| > 1$ was studied by Pelyukh and Sharkovskii in [18]. They gave a representation of general solution of the equation when $|c| > 1$. In [8] were given some results on the existence of bounded and finite solutions of equations with linearly transformed argument. Paper [2] presents majorants for solutions of the equation. In [3] were given some sufficient conditions under which zero solution of the equation is asymptotically stable.

Motivated by this line of investigations and our idea of studying complex mathematical models, frequently based on some iterations (see, e.g., [14,23–30]), here we study stability of the zero solution of the following functional-differential equation

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http://dx.doi.org/10.1016/j.amc.2014.04.010
0096-3003/

$$x'(t) = ax(t) + bx(\tau_0(t)) + \sum_{j=1}^{k} c_j x'(\tau_j(t)) + f(x(t), x(\tau_{k+1}(t)), \ldots, x(\tau_{k+l}(t))) \tag{2}$$

on the interval $[1, +\infty)$, where $a, b, c_j, j = \overline{1,k}$ are real numbers, $f : \mathbb{R}^{l+1} \to \mathbb{R}$ is a continuous function, and $\tau_j(t), j = \overline{0, k+l}$ are functions in the space $C^2[0, \infty)$ such that

$$\tau_j(0) = 0, \quad j = \overline{0, k+l}, \tag{3}$$

$$0 < \inf_{t \geqslant 0} \tau_j'(t) \leqslant \sup_{t \geqslant 0} \tau_j'(t) < 1, \quad j = \overline{0, k+l} \tag{4}$$

and that the function

$$\tau(t) := \min_{j=\overline{0,k+l}} \tau_j(t) \tag{5}$$

is a C^1 function on the interval $\mathbb{R}_+ := [0, +\infty)$.

2. Main result

In this section we formulate and prove the main result in this note.

Theorem 1. *Assume that the following conditions hold:*

(a) $a < 0, b, c_j \in \mathbb{R}, j = \overline{1,k}$;
(b) $f : \mathbb{R}^{l+1} \to \mathbb{R}$ *is a continuous function such that* $f(0, \ldots, 0) = 0$ *and*

$$|f(x_1, \ldots, x_{l+1}) - f(y_1, \ldots, y_{l+1})| \leqslant \sum_{j=1}^{l+1} L_j(\sigma)|x_j - y_j| \tag{6}$$

for every $x_j, y_j, j = \overline{1, l+1}$, *such that* $\max_{j=\overline{1,l+1}}\{|x_j|, |y_j|\} \leqslant \sigma$, *where the functions* $L_j(\sigma), j = \overline{1, l+1}$, *are defined on* $[0, +\infty)$ *and such that* $L_j(\sigma) \to 0$ *as* $\sigma \to 0, j = \overline{1, l+1}$;
(c) *functions* $\tau_j(t), j = \overline{0, k+l}$, *are* C^2 *functions satisfying conditions (3) and (4), and* $\tau(t)$ *defined in (5) is a* C^1 *function;*
(d) *the following two quantities*

$$\beta := \sup_{t \geqslant 1} \left(|b| + \sum_{j=1}^{k} \left| \frac{ac_j}{\tau_j'(t)} + \frac{c_j \tau_j''(t)}{\left(\tau_j'(t)\right)^2} \right| \right),$$

$$\gamma := \sup_{t \geqslant 1} \sum_{j=1}^{k} \left| \frac{c_j}{\tau_j'(t)} \right|, \tag{7}$$

satisfy the next condition

$$\gamma + \frac{\beta}{|a|} < 1. \tag{8}$$

Then the zero solution of equation (2) is asymptotically stable.

Proof. Multiplying Eq. (2) by e^{-at} we obtain

$$(x(t)e^{-at})' = \left(bx(\tau_0(t)) + \sum_{j=1}^{k} c_j x'(\tau_j(t)) + f(x(t), x(\tau_{k+1}(t)), \ldots, x(\tau_{k+l}(t))) \right) e^{-at}.$$

By integrating this equality from 1 to t, it follows that

$$x(t) = e^{at} \left(x(1)e^{-a} + \int_1^t e^{-as} \left(bx(\tau_0(s)) + \sum_{j=1}^{k} c_j x'(\tau_j(s)) + f(x(s), x(\tau_{k+1}(s)), \ldots, x(\tau_{k+l}(s))) \right) ds \right). \tag{9}$$

By using the integration in parts we have that

$$\int_1^t e^{-as} x'(\tau_j(s)) ds = e^{-as} \frac{x(\tau_j(s))}{\tau_j'(s)} \Bigg|_1^t + \int_1^t e^{-as} \left(\frac{a}{\tau_j'(s)} + \frac{\tau_j''(s)}{\left(\tau_j'(s)\right)^2} \right) x(\tau_j(s)) ds \tag{10}$$