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Modified spectral parameter power series representations for solutions of Sturm–Liouville equations and their applications ☆

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ABSTRACT

Spectral parameter power series (SPPS) representations for solutions of Sturm–Liouville equations proved to be an efficient practical tool for solving corresponding spectral and scattering problems. They are based on a computation of recursive integrals, sometimes called formal powers. In this paper new relations between the formal powers are presented which considerably improve and extend the application of the SPPS method. For example, originally the SPPS method at a first step required to construct a nonvanishing (in general, a complex-valued) particular solution corresponding to the zero-value of the spectral parameter. The obtained relations remove this limitation. Additionally, equations with "nasty" Sturm–Liouville coefficients $1/p$ or r can be solved by the SPPS method.

We develop the SPPS representations for solutions of Sturm–Liouville equations of the form

$$(p(x)u')' + q(x)u = \sum_{k=1}^{N} \lambda^k R_k[u], \quad x \in (a,b)$$

where $R_k[u] := r_k(x)u + s_k(x)u'$, $k = 1, \dots N$, the complex-valued functions p, q, r_k, s_k are continuous on the finite segment $[a,b]$.

Several numerical examples illustrate the efficiency of the method and its wide applicability.

1. Introduction

Solutions of sufficiently regular linear second order Sturm–Liouville equations considered as functions of a spectral parameter are entire functions which in particular means that they admit a normally convergent Taylor series representation in terms of the spectral parameter in the whole complex plane. The coefficients of the series are functions of the independent variable. For example, in the simplest case of the equation $y''(x) = \lambda y(x)$ two linearly independent solutions (satisfying in the origin the initial conditions $(1,0), (0,1)$) can be chosen in the form $y_1(x) = \cosh\sqrt{\lambda}x$ and $y_2(x) = \left(\sinh\sqrt{\lambda}x\right)/\sqrt{\lambda}$. The Taylor coefficients in their power series in terms of the spectral parameter λ with the center $\lambda = 0$ are powers of the independent variable divided by corresponding factorials $x^{2n}/(2n)!$ and $x^{2n+1}/(2n+1)!$ respectively.

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In [17] a simple way for calculating the Taylor coefficients for spectral parameter power series (SPPS) defining solutions of the Sturm–Liouville equation $(pu')' + qu = \lambda u$ was proposed, based on the theory of complex pseudoanalytic functions. In [19] (see also [18]) that result was extended onto equations of the form

$$(pu')' + qu = \lambda ru \tag{1.1}$$

and proved in a simpler way with no need of pseudoanalytic function theory (see Theorem 2.1 below). The Taylor coefficients in the SPPS representations are calculated as recursive integrals and called formal powers. The SPPS representations found numerous applications, see two recent review papers [15,20]. In [14] SPPS representations were obtained for solutions of fourth order Sturm–Liouville equations of the form

$$(pu'')'' + (qu')' = \lambda R[u]$$

where R is a linear differential operator of the order $n \le 3$, and in [11] for Bessel-type singular Sturm–Liouville equations. In [22] the SPPS representations were obtained for equations of the form

$$(p(x)u')' + q(x)u = \sum_{k=1}^{N} \lambda^k r_k(x)u$$

and used for studying spectral problems for Zakharov–Shabat systems.

In [9] it was shown that at least in the case of the one-dimensional Schrödinger equation

$$u'' + qu = \lambda u \tag{1.2}$$

the formal powers are the images of usual powers x^k, $k = 0, 1, 2, \ldots$ under the action of a corresponding transmutation operator. In [21] based on this observation a new method for solving spectral problems for (1.2) was developed. The method possesses a remarkable unique feature: it allows one to compute thousands of eigendata with a non-decreasing accuracy. In [10,8,9,16] methods for solving different problems for partial differential equations involving the computation of formal powers were developed.

Thus, the computation of formal powers is required for application of different methods and in different models. An important restriction for computing formal powers as proposed in [17,19] and further publications consisted in the necessity of a nonvanishing particular solution of the equation

$$(pv')' + qv = 0. \tag{1.3}$$

When p and q are real valued (and sufficiently regular) such nonvanishing solution can be proposed in the form $v_0 = v_1 + iv_2$ where v_1 and v_2 are arbitrary linearly independent real-valued solutions of (1.3). However for complex-valued coefficients p and q there is no such simple way for its construction. Moreover, even when v_0 does not vanish but in some points is relatively close to zero, the computation of formal powers may present difficulties.

In the present work we solve two problems. (1) We develop an SPPS representation which is not limited to nonvanishing particular solutions of auxiliary equations and admits certain "nastiness" in the coefficients. For example, p is allowed to have zeros. (2) We extend the SPPS method to equations of the form

$$(p(x)u')' + q(x)u = \sum_{k=1}^{N} \lambda^k R_k[u], \quad x \in (a, b) \tag{1.4}$$

where $R_k[u] := r_k(x)u + s_k(x)u'$, $k = 1, \ldots N$, the complex-valued functions p, q, r_k, s_k are continuous on the finite segment $[a, b]$. The presented numerical results show that nowadays this is one of the most accurate ways for solving corresponding spectral problems with a wide range of applicability (e.g., few available algorithms are applicable to complex coefficients, complex spectra, polynomial pencils of operators, etc.).

In Section 2 we prove new relations concerning formal powers and obtain the modified SPPS representations for Sturm–Liouville equations of the form (1.1). In Section 3 we extend this result to equations of the form (1.4). In Section 4 we describe the algorithm and the numerical implementation of the proposed method for solving spectral problems and give eight numerical examples illustrating its performance.

2. SPPS representations

2.1. The original SPPS representation

In [19] the following theorem was proved.

Theorem 2.1 (SPPS representation, [19]). *Assume that on a finite segment* $[a, b]$, *equation*

$$(pv')' + qv = 0, \tag{2.1}$$