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Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations

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ABSTRACT

Within the framework of the theory of the column and row determinants, we obtain determinantal representations of the Drazin inverse both for Hermitian and arbitrary matrices over the quaternion skew field. Using the obtained determinantal representations of the Drazin inverse we get explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of a quaternion matrix equation $\mathbf{AXB} = \mathbf{D}$ and consequently $\mathbf{AX} = \mathbf{D}$, and $\mathbf{XB} = \mathbf{D}$ in two cases if $\mathbf{A}$, $\mathbf{B}$ are Hermitian or arbitrary.

1. Introduction

Throughout the paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k | i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by $\mathbf{I}$. Let $\mathrm{M}(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, the symbols $\mathbf{A}^*$ stands for the conjugate transpose (Hermitian adjoint) matrix of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

As one of the important types of generalized inverses of matrices, the Drazin inverses and their applications have well been examined in the literature (see, e.g., [1–6]). Stanimirovič and Djordjević in [7] have introduced a determinantal representation of the Drazin inverse of a complex matrix based on its full-rank representation. In [8,9] we obtain determinantal representations of the Drazin inverse of a complex matrix used its limit representation. It allowed to obtain the analogs of Cramer's rule for the Drazin inverse solutions of the following matrix equations

$$\mathbf{AX} = \mathbf{D}, \tag{1}$$

$$\mathbf{XB} = \mathbf{D}, \tag{2}$$

$$\mathbf{AXB} = \mathbf{D}. \tag{3}$$

This paper partially extends studies conducted in [8,9] from the complex field to the quaternion skew field. In the case of quaternion matrices we are faced with the problem of the determinant of a square quaternion matrix. But recently in [10,11] the theory of the column and row determinants of a quaternion matrix have been developed. Within the framework of the

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theory of the column and row determinants, determinantal representations of the Moore–Penrose inverse by analogs of the classical adjoint matrix in [12] and analogs of Cramer's rule for the least squares solutions with minimum norm of the matrix equations (1)–(3) in [13] have been obtained. (The complex case of analogs of Cramer's rule for the least squares solutions with minimum norm of the matrix equation (1)–(3) have been considered in [14].)

In [15–17] the authors have received determinantal representations of the generalized inverses $\mathbf{A}^2_{T_{T_1},S_1}$, $\mathbf{A}^2_{l_{T_2},S_2}$, and $\mathbf{A}^2_{(T_1,T_2),(S_1,S_2)}$, and consequently of the Moore–Penrose and Drazin inverses over the quaternion skew field by the theory of the column and row determinants as well. But in obtaining of these determinantal representations another auxiliary matrices together with $\mathbf{A}$ are used.

In this paper we aim to obtain determinantal representations of the Drazin inverse both for Hermitian and arbitrary matrices by using only their entries or entries of their powers and explicit representation formulas (analogs of Cramer's rule) for the Drazin inverse solutions of quaternion matrix equations (1)–(3), where $\mathbf{A}$ and $\mathbf{B}$ are both Hermitian or arbitrary, without any restriction.

The paper is organized as follows. We start with some basic concepts and results from the theory of the row and column determinants and the theory on eigenvalues of quaternion matrices in Section 2. In Section 3, we give the determinantal representations of the Drazin inverse for a Hermitian quaternion matrix in Section 3.1 and an arbitrary quaternion matrix in Section 3.2 In Section 4, we obtain explicit representation formulas for the Drazin inverse solutions of quaternion matrix equations (3), and consequently (1) and (2). In Section 4.1 $\mathbf{A}$, $\mathbf{B}$ are Hermitian and in Section 4.1 they are arbitrary. In Section 5, we show a numerical example to illustrate the main result.

2. Elements of the theory of the column and row determinants

Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 2.1. The ith row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined for all $i = 1, \ldots, n$ by putting

$$\mathrm{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i} \ldots a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}},$$

$$\sigma = \left(i i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}\right) \ldots \left(i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}\right),$$

with conditions $i_{k_2} < i_{k_3} < \ldots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for $t = \overline{2,r}$ and $s = \overline{1,l_t}$.

Definition 2.2. The jth column determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined for all $j = 1, \ldots, n$ by putting

$$\mathrm{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} i_{k_r}} \ldots a_{j j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

$$\tau = \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right)\left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right),$$

with conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = \overline{2,r}$ and $s = \overline{1,l_t}$.

Suppose $\mathbf{A}^{ij}$ denotes the submatrix of $\mathbf{A}$ obtained by deleting both the ith row and the jth column. Let $\mathbf{a}_{.j}$ be the jth column and $\mathbf{a}_{i.}$ be the ith row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its jth column with the column $\mathbf{b}$, and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$.

We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n, j \in J_n$ and $I_n = J_n = \{1, \ldots, n\}$.

Proposition 2.1 [10]. *If* $b \in \mathbb{H}$, *then* $\mathrm{rdet}_i \mathbf{A}_{i.}(b \cdot \mathbf{a}_{i.}) = b \cdot \mathrm{rdet}_i \mathbf{A}$ *for all* $i = 1, \ldots, n$.

Proposition 2.2 [10]. *If* $b \in \mathbb{H}$, *then* $\mathrm{cdet}_j \mathbf{A}_{.j}(\mathbf{a}_{.j} \cdot b) = \mathrm{cdet}_j \mathbf{A} \cdot b$ *for all* $j = 1, \ldots, n$.

Proposition 2.3 [10]. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in I_n$ *such that* $a_{tj} = b_j + c_j$ *for all* $j = 1, \ldots, n$, *then*

$$\mathrm{rdet}_i\, \mathbf{A} = \mathrm{rdet}_i\, \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{rdet}_i\, \mathbf{A}_{t.}(\mathbf{c}),$$
$$\mathrm{cdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}_{t.}(\mathbf{b}) + \mathrm{cdet}_i\, \mathbf{A}_{t.}(\mathbf{c}),$$

where $\mathbf{b} = (b_1, \ldots, b_n), \mathbf{c} = (c_1, \ldots, c_n)$ *and for all* $i = 1, \ldots, n$.

Proposition 2.4 [10]. *If for* $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ *there exists* $t \in J_n$ *such that* $a_{it} = b_i + c_i$ *for all* $i = 1, \ldots, n$, *then*