



Closed form numerical solutions of variable coefficient linear second-order elliptic problems

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ABSTRACT

In this work we develop an alternative numerical technique which allows to construct a numerical solution in closed form of variable coefficient linear second-order elliptic problems with Dirichlet boundary conditions. The elliptic partial differential equation is approximated by a consistent explicit difference scheme and using a discrete separation of the variables method we determine a closed form solution of the two resulting discrete boundary value problems with the separated variables, avoiding to have to solve large algebraic systems. One of these boundary value problems is a discrete Sturm–Liouville problem which guarantees the qualitative properties of the exact solution of elliptic problem. A constructive procedure for the computation of the numerical solution is given and an illustrative example is included.

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1. Introduction

Elliptic partial differential equations arise usually from equilibrium or steady-state fluid flow and heat problems and their solutions, in relation to the calculus of variations, frequently maximize or minimize an integral representing the energy of the system. Exact solutions only exists for a few special cases with simple geometries and boundary conditions, or for simplified constants coefficients equations, in which some of the more complicated physical phenomena are neglected. Fortunately, numerical analysis in these equations can offer reliable solutions.

Apart from some techniques such as meshless methods [1,2] and those based on particular transformations used to solve special problems [3,4], the most used are related mesh methods as the finite difference method [5–7], the finite-volume method [8,9] and the finite element method [10,11].

In this paper we consider an explicit finite difference scheme for the following linear second-order homogeneous elliptic problem with Dirichlet boundary conditions

$$r(x)u_{yy}(x, y) + [p(x)u_x(x, y)]_x - q(x)u(x, y) = 0, \quad a < x < b, \quad c < y < d, \quad (1)$$

$$u(a, y) = f_1(y), \quad y \in [c, d], \quad (2)$$

$$u(b, y) = g_1(y), \quad y \in [c, d], \quad (3)$$

$$u(x, c) = f_2(x), \quad x \in [a, b], \quad (4)$$

$$u(x, d) = g_2(x), \quad x \in [a, b], \quad (5)$$

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where $r(x)$, $p(x)$, $q(x)$, $f_1(y)$, $f_2(x)$, $g_1(y)$, $g_2(x)$ are continuous real functions and

$$\left. \begin{array}{l} r(x) > 0, \quad p(x) > 0, \quad q(x) > 0, \quad x \in [a, b] \\ p(x) \text{ is differentiable} \end{array} \right\}. \quad (6)$$

Discretization of the partial differential equation (pde) together with the boundary conditions give rise to an algebraic discretized problem where the unknowns are the numerical values of the solution at the mesh points. This algebraic problem can be compactly written as a linear system $Au = b$, where the entries of the matrix A and the vector b involve the structure of the pde and the boundary value conditions. A detailed study of this algebraic treatment may be found in [12,13, Chapter 5].

An alternative approach to solve the discretized problem as a mere algebraic system, that at the same time tries to preserve the properties of the continuous eigenfunction method for the continuous problem [14], is based on the construction of a discrete separation of the variables method for the resulting discretized problem. This method has been successfully used in [15–17] for solving parabolic and hyperbolic problems, and it is considered here for the solution of the elliptic problem (1)–(5) providing a closed form numerical solution.

For the study of the elliptic problem (1)–(5) we decompose the boundary conditions (2)–(5) in the four following cases:

$$\begin{aligned} \text{(I)} \quad & \left. \begin{array}{l} u(a, y) = 0, \quad y \in [c, d] \\ u(b, y) = 0, \quad y \in [c, d] \\ u(x, c) = f_2(x), \quad x \in [a, b] \\ u(x, d) = 0, \quad x \in [a, b] \end{array} \right\}, \quad \text{(II)} \quad \left. \begin{array}{l} u(a, y) = 0, \quad y \in [c, d] \\ u(b, y) = 0, \quad y \in [c, d] \\ u(x, c) = 0, \quad x \in [a, b] \\ u(x, d) = g_2(x), \quad x \in [a, b] \end{array} \right\}, \\ \text{(III)} \quad & \left. \begin{array}{l} u(a, y) = f_1(y), \quad y \in [c, d] \\ u(b, y) = 0, \quad y \in [c, d] \\ u(x, c) = 0, \quad x \in [a, b] \\ u(x, d) = 0, \quad x \in [a, b] \end{array} \right\}, \quad \text{(IV)} \quad \left. \begin{array}{l} u(a, y) = 0, \quad y \in [c, d] \\ u(b, y) = g_1(y), \quad y \in [c, d] \\ u(x, c) = 0, \quad x \in [a, b] \\ u(x, d) = 0, \quad x \in [a, b] \end{array} \right\}. \end{aligned}$$

Potential advantages of the proposed method are that being explicit and based on a discrete eigenfunction method, the properties of the exact theoretical solution are preserved by the numerical approximation allowing that with just a few nodes the approximation is very good together with a low computational cost.

This paper is organized as follow. Section 2 deals with the discretization of Eq. (1) and the study of the consistency of the constructed numerical scheme. In Section 3 we construct numerical solutions considering cases I and II by means of a discrete separation of variables method. The similarity of the first two homogeneous boundary conditions allows to use the same discrete Sturm–Liouville problem in both cases whose eigenfunctions are determined throughout the eigenpairs of an algebraic eigenvalue problem. Cases III and IV are treated in Section 4 because both problems have the same last two homogeneous boundary conditions and the underlying discrete Sturm–Liouville problem. In these cases the eigenfunctions are analytically determined. Section 5 includes a constructive numerical algorithm of the original problem. We also include an illustrative numerical example. Section 6 summarizes the main conclusions of the paper.

2. Discretization and consistency

Let us begin this section by subdividing the plane domain $[a, b] \times [c, d]$ into a rectangular mesh of equal rectangles of sides $\Delta x = h$, $\Delta y = k$. Let N and M be natural numbers and let α be positive real number such that

$$h = \frac{b-a}{N+1}, \quad k = \frac{d-c}{M+1}, \quad \alpha = \frac{k}{h}. \quad (7)$$

Then a typical mesh point (x_i, y_j) verifies

$$\begin{aligned} x_i &= a + ih, \quad 0 \leq i \leq N+1, \\ y_j &= c + jk, \quad 0 \leq j \leq M+1. \end{aligned}$$

Let us denote $u(x_i, y_j)$, $r(x_i)$, $p(x_i)$, $q(x_i)$, $f_1(y_j)$, $f_2(x_i)$, $g_1(y_j)$ and $g_2(x_i)$, by $U(i, j)$, $r(i)$, $p(i)$, $q(i)$, $f_1(j)$, $f_2(i)$, $g_1(j)$ and $g_2(i)$ respectively. We approximate the derivatives of elliptic Eq. (1) by the following finite differences of second-order

$$u_{yy}(x_i, y_j) \approx \frac{U(i, j+1) - 2U(i, j) + U(i, j-1))}{k^2}, \quad (8)$$

$$[p(x_i)u_x(x_i, y_j)]_x \approx \frac{p(i)[U(i+1, j) - U(i, j)] - p(i-1)[U(i, j) - U(i-1, j)]}{h^2}, \quad (9)$$

obtaining the explicit difference scheme

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