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Multiple limit cycles and centers on center manifolds for Lorenz system

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ABSTRACT

For Lorenz system we investigate multiple Hopf bifurcation and center-focus problem of its equilibria. By applying the method of symbolic computation, we obtain the first three singular point quantities. It is proven that Lorenz system can generate 3 small limit cycles from each of the two symmetric equilibria. Furthermore, the center conditions are found and as weak foci the highest order is proved to be the third, thus we obtain at most 6 small limit cycles from the symmetric equilibria via Hopf bifurcation. At the same time, we realize also that though the same for the related three-dimensional chaotic systems, Lorenz system differs in Hopf bifurcation greatly from the Chen system and Lü system.

1. Introduction

In this paper, we consider Lorenz system which is taken the following form

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = cx_1 - x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{1}$$

where $a, b, c \in \mathbb{R}$. It is well-known, since Lorenz found the classical chaotic attractor in 1963, chaos become more and more attractive theoretical subject, especially in the nonlinear science. In the past decades, extensive investigations on chaotic systems have been carried out, particularly, for Lorenz system (see [1–5] and the references therein). Nevertheless, some dynamics properties of Lorenz system have not been completely understood by mathematicians, for example, the multiple Hopf bifurcation. Usually the single Hopf bifurcations of chaotic systems can be seen in many works, yet for the multiple one there exist only a few papers, as we know, Mello and Coelho [6] for Lü system, Messias et al. [7] for Chua's system, Wang and Huang [8] for Chen system. Here we investigate this problem of Lorenz system. At the same time, we also discuss the center-focus problem for the flow of Lorenz system restricted to the center manifold, which closely relates to the maximum number of limit cycles bifurcating from the equilibria.

For the center-focus problem on the center manifold of a system in $\mathbb{R}^3$, the authors of [9] fully studied the center-focus determination method in terms of an inverse Jacobi multiplier. The authors of [10,11] gave respectively the formal series method and the formal first integral method of determining the existence of a center. In addition, about this problem we

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also see [12,13]. In our process, mainly the method given by Wang et al. [10] is applied to compute the singular point quantities of the equilibria for Lorenz system, which has been proved to be algebraic equivalent to the corresponding focal values. In contrast to the more usual ones such as Liapunov functions-Poincaré normal form and integral averaging method (see [14]), it is convenient to compute the higher order focal values and solve the center-focus problem of the equilibrium (the more details can be seen in [15]).

The rest of this paper is organized as follows. In Section 2, the corresponding singular point quantities are computed and the center conditions on the center manifold are determined. In Section 3, the multiple Hopf bifurcations at the two symmetrical equilibria for Lorenz system are investigated, six limit cycles from them are obtained, and it is proved at most 6 small limit cycles from them via Hopf bifurcation. The results are not only identical with and complementary to the previous work on Hopf bifurcation in Lorenz system, but also helpful to compare the three related chaotic systems: Lorenz system, Chen system and Lü system.

2. Singular point quantities and center conditions

In this part, we investigate the singular point quantities of the corresponding equilibria. Evidently, Lorenz system (1) always has the equilibrium $O(0,0,0)$. Suppose that $b(c-1)>0$ holds, for system (1) there exist other two fixed points $E_1=(\sqrt{b(c-1)},\sqrt{b(c-1)},c-1)$ and $E_2=(-\sqrt{b(c-1)},-\sqrt{b(c-1)},c-1)$. The equations in (1) are invariant under the transformation:

$$(x_1,x_2,x_3)\mapsto(-x_1,-x_2,x_3), \tag{2}$$

which means that if $(x_1(t),x_2(t),x_3(t))$ is a solution, then $(-x_1(t),-x_2(t),x_3(t))$ is a solution too. Therefore for the two symmetric equilibria we choose only E_1 to analyze in detail, and for O, we will also not give the detailed process which is similar to the case of E_1.

Now we should consider the fixed point E_1 with the property: there is a pair of purely imaginary eigenvalues, that is, the equilibrium can undergo a generic Hopf bifurcation. Firstly, we notice the Jacobian matrix of system (1) at the point E_1 as follows

$$A=\begin{pmatrix} -a & a & 0\\ 1 & -1 & -\sqrt{b(c-1)}\\ \sqrt{b(c-1)} & \sqrt{b(c-1)} & -b\end{pmatrix}.$$

where $b(c-1)>0$. For convenience of calculation, we transform the equilibrium E_1 to the origin, and set $\mathbf{x}=(x_1,x_2,x_3)=\left(\tilde{x}_1+\sqrt{b(c-1)},\tilde{x}_2+\sqrt{b(c-1)},\tilde{x}_3+c-1\right)$, then system (1) takes the form

$$\dot{\mathbf{x}}=A\begin{pmatrix} x_1+\sqrt{b(c-1)}\\ x_2+\sqrt{b(c-1)}\\ x_3+c-1\end{pmatrix}+\begin{pmatrix}0\\ -(x_1+\sqrt{b(c-1)})(x_3+c-1)\\ (x_1+\sqrt{b(c-1)})(x_2+\sqrt{b(c-1)})\end{pmatrix}. \tag{3}$$

Here we use x_i instead of $\tilde{x}_i$ for $i=1,2,3$. To satisfy the necessary eigenvalue conditions that A has a pair of purely imaginary eigenvalues $\pm\mathbf{i}\omega(\omega>0)$, we let $\mathrm{Det}(A)=(A_{11}+A_{11}+A_{33})\mathrm{tr}(A)$, i.e. $c(a-b-1)=a(a+b+3)$ and $\omega^2=b(a+c)$, then

$$\begin{aligned} &b=(a-1)\omega^2/(2a+2a^2+\omega^2),\\ &c=(a^2+3a+\omega^2)/(a-1). \end{aligned} \tag{4}$$

Thus one can construct a matrix P which transforms A to be a block-diagonal one, i.e. using the nondegenerate transformation $\mathbf{x}=P\mathbf{y}$, such that

$$P^{-1}AP=\begin{pmatrix} 0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & -2aD_1/D_2\end{pmatrix}.$$

where

$$P=\begin{pmatrix} \frac{a(a+1)}{\sqrt{D_1D_2}} & \frac{a\omega}{\sqrt{D_1}} & \frac{D_2\sqrt{D_2}}{2(1-a)\omega\sqrt{D_1}}\\ \frac{a+a^2+\omega^2}{\sqrt{D_1D_2}} & \frac{-\omega}{\sqrt{D_1D_2}} & \frac{(2+2a+\omega^2)\sqrt{D_2}}{2(a-1)\omega\sqrt{D_1}}\\ 0 & 1 & 1\end{pmatrix}.$$

and $D_1=(1+a)^2+\omega^2,\ D_2=2a(1+a)+\omega^2$. Moreover, after a time scaling: $t\ \to\ t/\omega$, we can get a new system from the system (3):