A comparative study of deterministic and stochastic dynamics for a non-autonomous allelopathic phytoplankton model

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ABSTRACT

In this paper, we investigate a non-autonomous competitive phytoplankton model with periodic coefficients in deterministic and stochastic environment, respectively. We prove the existence of at least one positive periodic solution together with it's global asymptotic stability. The existence of periodic solution has been obtained by using the continuation theorem of coincidence degree theory proposed by Gaines and Mawhin. We formulate the corresponding stochastic model by perturbing the growth rate parameters by white noise terms. We prove that all the higher order moments of the solution to the stochastic system is uniformly bounded which ensure that the solution of the stochastic system is stochastically bounded. We provide easily verifiable sufficient conditions for non-persistence in mean, extinction and stochastic permanence of the stochastic system. Sufficient condition for permanence shows that if the noise intensity is very low then the solution of the stochastic system persists in the periodic coexistence domain of the deterministic system. We perform exhaustive numerical simulations to validate our analytical findings.

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1. Introduction

In this paper, we investigate the following non-autonomous Lotka–Volterra type competition model of two interacting phytoplankton species in which each species produces toxic chemical which affect negatively the growth of other species:

$$\frac{du(t)}{dt} = u(t)(k_1(t) - \alpha_1(t)u(t) - \beta_1(t)v(t) - \gamma_1(t)u(t)v(t))$$
$$\frac{dv(t)}{dt} = v(t)(k_2(t) - \alpha_2(t)v(t) - \beta_2(t)u(t) - \gamma_2(t)u(t)v(t))$$
$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \quad t \in [0, \infty). \tag{1}$$

Here $u(t)$, $v(t)$ are the densities of two competitive phytoplankton species at any instant of time t. The coefficients $k_1(t)$, $k_2(t)$ are the cell proliferation rates, $\alpha_1(t)$, $\alpha_2(t)$ are the intra specific competition rates, $\beta_1(t)$, $\beta_2(t)$ are the rates of inter-specific competition and $\gamma_1(t)$, $\gamma_2(t)$ are the rate of allelopathic chemical released by the second and by the first

phytoplankton species respectively. Here we assume that $k_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, $i = 1, 2$ are all continuous and bounded functions of time for all $t \geqslant 0$.

The autonomous model corresponding to the model (1) was first proposed by Maynard-Smith [1] which accounts for the allelopathic interaction between two competing species. Preliminary stability analysis of the model (1) with respect to the context of phytoplankton allelopathy was proposed by Chattopadhyay [2] and in here, the allelopathic interaction can strengthen the stability of coexistence steady-state is shown. In reality, the introduction of allelopathic interaction into the competition model can result in more complicated dynamics including the presence of two co-existing steady-state and their stability depends upon the magnitudes of the parameters. Interestingly, the proposed model can exhibit the bi-stable scenario for a range of parameter values and the coexistence of the competing species depend upon the initial population densities. A through analysis of the model proposed by Maynard-Smith is recently investigated by Gupta et al. [3] with a few modifications for a different context. Dynamics of model (1) with discrete time delay and almost periodic coefficients are investigated by Abbas et al. [4]. In this work, the role of time delay and environmental quasi-periodicity on the bloom formation by two competing phytoplankton species are thoroughly investigated.

Several authors [5–13] have considered the non-autonomous models of interacting species, which is capable to take care of seasonal variation in environmental conditions. The concerned models are of Lotka–Volterra type but without allelopathic interaction term ($\gamma_i = 0$). Various factors (intensity of sunlight, temperature, salinity of water, amount of soluble nutrients, etc.) in the environment varies seasonally, in a periodic manner, and affect the intrinsic parameters involved the mathematical models of interacting populations. Apart from these, the assumption behind periodic oscillation of the system parameters are justified due to other seasonal factors like mating habits, availability of food etc. Hence it is quite reasonable to study the non-autonomous models of ecological systems driven by periodic external forces. Many researchers have studied the non-autonomous ecological models where intrinsic rates are considered as bounded periodic functions of time to model the seasonal variability [11,10,8,14–16]. Motivated by this fact, in this work, we consider the deterministic model where the parameters of the system (1) are periodic with common period, the second assumption is due to the simplicity of mathematical calculations.

The main problem in the study of population growth model with periodic coefficients is the existence of positive periodic solution and its global stability. Hence it is reasonable to search for conditions under which the concerned non-autonomous system with periodic coefficients will have a positive periodic solution which is globally asymptotically stable. In this context, we assume that the parameters in the system (1) are periodic in t with a fixed period $T_1 > 0$ and will derive the parametric restriction in terms of the bounds of the periodic coefficients for the existence of positive periodic solution and its global asymptotic stability for the model (1).

The Lotka–Volterra formalism to the models of interacting species are important to study the multi-species population dynamics. The most important issues to study for these type of systems involved in mathematical ecology are the persistence, extinctions, global asymptotic behaviors and existence of coexistence state(s) (e.g. the positive equilibrium level, positive periodic solution, etc.) [5,6,10,11,8,9]. There have been numerous works for the non-autonomous Lotka–Volterra type systems of interacting species. In [5], Ahmad studied a two-dimensional non-autonomous model of two competing species and showed the existence of almost periodic solutions which is globally asymptotically stable under some parametric restrictions. Moreover in another paper [6], Ahmad has shown that if the coefficients of the two-dimensional non-autonomous model of two competing species are continuous and satisfy certain parametric restrictions, then any solution that is positive at some point has the property that one of its components vanishes while the other approaches to a solution of logistic equation. This phenomena is called the principle of competitive exclusion. Gopalsamy [10] had studied the existence and global asymptotic stability of periodic solutions for non-autonomous systems under the assumption that all the coefficients are positive and periodic. He also concluded that under certain condition (see Theorem 3.1 in [10]), no complex behavior is possible for the system considered in [10]. Alvarez and Lazer [11] extended the result considered earlier by Ahmad [5] and reported the existence of T periodic solution of the model, using Mahwin degree theory. Cushing [8] studied the effect of periodicity of the solutions of the model system considered in [5] by using bifurcation techniques. Li and Chen [9] have considered the non-autonomous model (1) and established that the extinction of one species will imply the stabilization of other species at some steady-state value. Motivated by the above works, in this paper we are interested to study the system (1) and establish a series of sufficient conditions on the permanence, existence of positive periodic solution and global asymptotic stability of periodic solution for the model (1).

Apart from the successful investigation of the deterministic systems, several researchers are interested about the fact that the extension of the deterministic non-autonomous models to a stochastic system as the population growth is significantly affected by the environmental driving forces, which is an important component in many interacting population system [17,18]. Since the parameters involved in the deterministic system are unable to capture the random environmental fluctuations, deterministic models have some limitations in mathematical modeling of ecological systems. May [19] pointed out that due to environmental driving forces, birth rates, carrying capacities, competition coefficients and other intrinsic parameters involved with the system show certain amount of random fluctuation due to stochastic fluctuation in the environmental conditions. Such kind of fluctuations can be modeled by introducing white noise as well as by colored noise terms [20]. It is well known that if the colored noise is not strongly correlated, then we can approximate the colored noise by white noise and this approximation works very well. Hence most of the authors introduced the white noise term(s) into their modelling approach to study the effect of the environmental fluctuation on the dynamics of interacting populations [21–29]. Motivated by this fact, we extend our deterministic model (1) to a stochastic model by introducing multiplicative white noise terms. As