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Mean action time as a measure for fin performance in one dimensional fins of exponential profiles

I. Rusagara, C. Harley *

Centre for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

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ABSTRACT

The aim of this paper is to numerically solve the one dimensional time dependent heat transfer equation while considering different types of exponential profiles. We consider non-linear thermal conductivity via the power law and employ a numerical relaxation scheme due to its simplicity. Furthermore, we obtain the mean action time and propose it as a novel means of analysing the fin performance. Our results are validated through comparing with benchmark results and our new index for fin performance is shown to provide meaningful insight.

1. Introduction

There exists a wide range of applications for extended surfaces called fins as has been discussed by several researchers [1–4]. When one introduces notions such as a fin's profile, form, orientation, height, length and spacing in an array [5–10] into a model for heat transfer one finds that the equation becomes highly non-linear and stiff. In Rusagara and Harley [11] we considered singular profiles and presented an effective computational method for the solution of a similar model to the one considered here. In this research, while our equation remains highly non-linear and stiff, we instead focus on parametric fins with exponential profiles where the thermal conductivity is described by the power law. This choice for the profile is considered in a form which introduces a parameter allowing for a reduction to the rectangular profile. This is done to allow for a comparison of our numerical solutions to analytical solutions.

The analytical solutions of equations of the kind as considered in this work are not easily obtained, and may often only be found through a simplification of the model – see [4,11] for concise discussions regarding this and references therein. As such, given that we aim to investigate a highly non-linear partial differential equation, we will employ computational methods for the solution of the equation. There are various methods available for the solution of equations of the type discussed here – see [16–19] and discussions therein particularly pertaining to Spectral numerical integration which is shown to be highly effective. Given the literature – see for instance [13–15] – surrounding numerical relaxation schemes we turn to this methodology for the solution of the nonlinear partial differential equation under discussion. There are various motivations for this choice: (1) the method is able to effectively deal with nonlinear flux and source terms without requiring linearisation, (2) the scheme is simple to implement and (3) is able to achieve higher order accuracy in capturing weak solutions without using Riemann Solvers spatially or systems of algebraic equations temporally.

* Corresponding author.
E-mail addresses: Charis.Harley@wits.ac.za, Charis.Harley@gmail.com (C. Harley).

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Wang et al. [12] stated that a system of highly coupled equations with stiff non-linear source terms needs to be treated in the proper manner computationally so that one does not obtain spurious steady state numerical solutions – this is an appropriate consideration given the nature of the computational method we will employ. As such, we take care when discretising the system to maintain a discrete analogy for the zero relaxation limit which is consistent with the original equation. We consider the one-dimensional case, not due to its simplicity, but rather so that we are able to validate our results through a clear visualisation against benchmark results. An extension to higher dimensions is straight forward given the methodology employed in this work and in [11].

Due to its robustness, as discussed in [14], the scheme shall be applied to the transient heat equation under consideration and be further employed as a means of computing the mean action time for which there is no analytical technique available to us due to the nature of the partial differential equation under discussion. We are interested in the mean action time of the process given that in heat transfer the length of time taken by the process to reach a steady state is of import. In fact, we are interested in introducing a novel means of considering the fin performance, by relating it to the time taken by the process to reach some type of equilibrium. This proposed index is critical since the duration of a process plays a major role in deciding the proportions of a fin so as to be able to accelerate the process of heat transfer. However, as stated by Landman and McGuiness [20], diffusive processes often take an infinite amount of time to come to equilibrium and as such trying to obtain the time at which the process has reached equilibrium may not be possible either computationally (due to often incurred computational error) or analytically (due to the structure of the model). For this reason, we consider an approximation instead, i.e. instead of considering the time taken to reach a steady state we investigate the mean action time as a measure of the time taken to reach some type of equilibrium.

The structure of the problem under consideration does not lend itself towards obtaining the mean action time as done by McNabb and Wake [21] and McNabb [22]. In these works an approach is proposed which requires one to employ Poisson's equation such that the mean action time is easily computed without need of the original problem's solution [21]. In our case, we are unable to write the partial differential equation in conserved form due to the presence of a complex non-linear flux term and non-linear source term. As such we are unable to follow the work in [21] without certain alterations. In fact, due to this complication and based on the accuracy of relaxation schemes as stated by Jin and Xin [15], we investigate the mean action time via the methodology introduced in McNabb and Wake [21] but by computing the mean action time numerically via a relaxation scheme instead. In order to compute the mean action time numerically in this fashion, given that we require the equation to be in conserved form, we also assume that the thermo-geometric parameter $\mathcal{M}$ is small enough so that while it has an impact it does not dominate to the degree that the method of McNabb and Wake [21] is inappropriate. This approach employed is novel and allows us to propose that the mean action time can be used as a means of assessing the fin performance.

Our research has two components: firstly, we establish a numerical relaxation scheme and secondly we propose a direct computational approach for computing the mean action time for reaching a steady state for small values of the thermo-geometric parameter $\mathcal{M}$. Given that the mean action times have not as yet been obtained – to the best of the author's knowledge – by other researchers, we are unable to compare with other established results. We are able to justify our work given the methodology employed and via the results produced by our investigation: we show that the mean action times computed consistently provide $\frac{2}{3}$ of the steady state temperature determined computationally at time τ_{max} [20]. The motivation behind this research is the idea that steady state solutions in and of themselves are unable to exhaustively define the fin performance. In fact, we claim that the insight obtained from steady state solutions regarding fin performance is insufficient unless the time (or an approximation thereof) taken by the process to reach the given state can be provided by said solution.

2. Numerical relaxation scheme for one dimensional heat transfer

The relaxation scheme is structured via the introduction of a linear system with source term. We consider

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(f(x)k(\theta)\frac{\partial\theta}{\partial x}\right) - \mathcal{M}^2\theta^{n+1}, \quad 0 \le x \le 1, \quad \tau \geqslant 0, \tag{1}$$

where boundary conditions are as follows

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0 \quad \text{at the fin tip}, \tag{2}$$

$$\theta(\tau,\ 1) = 1 \quad \text{at the base of the fin}. \tag{3}$$

The initial condition is given as

$$\theta(0,\ x) = 0. \tag{4}$$

We define θ as the dimensionless temperature, τ the dimensionless time, x the dimensionless spatial variable, $f(x)$ the dimensionless fin profile and $k(\theta)$ the dimensionless thermal conductivity. Furthermore, the thermo-geometric parameter is defined as $\mathcal{M} = \frac{2Ph_bL^2}{k_a\delta_bA_p}$ with A_p the fin profile area, P the perimeter of the fin, L the length of the fin, δ_b the fin thickness at the base of the fin, h_b the heat transfer coefficient at the base and k_a the thermal conductivity of the fin at the ambient