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Exact closed form solution for the analysis of the transverse vibration modes of a Timoshenko beam with multiple concentrated masses

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A R T I C L E   I N F O

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A B S T R A C T

Concentrated masses on the beams have many industrial applications such as gears on a gearbox shafts, blades and disks on gas and steam turbine shafts, and mounting engines and motors on structures. Transverse vibration of the beam carrying a point mass was studied in many cases by both Euler–Bernoulli and Timoshenko beam theory for a limited number of concentrated masses mounted on a specific place on the beam. This was also investigated for a beam carrying multiple concentrated masses, yet they were solved by numerical methods such as Differential Quadrature (DQ) method. The present study investigated an exact solution for free transverse vibrations of a Timoshenko beam carrying multiple arbitrary concentrated masses anywhere on the beam with various boundary conditions. Using Dirac's delta in governing equations, the effects of concentrated masses were imposed. After extracting a closed form solution, basic functions were used to reduce the amount of computations. Standard symmetric and asymmetric boundary conditions were enforced for beam; in addition, the effects of value, position, and number of concentrated masses were examined. Generally, while the existence of concentrated masses reduces the natural frequencies, the reduction depends on the parameters of concentrated masses. Finally, there were acquired mode shapes for different boundary conditions and different value, position, and number of concentrated masses.

1. Introduction

Numerous studies were conducted for the analysis of transverse vibration of elastic beams which carry a restricted number of point masses on a limited position of beam with only one or two boundary conditions. Most of these studies were presented without considering the effects of shear forces and rotating inertia in beams. They utilized Euler–Bernoulli beam theory while Timoshenko beam theory has more accurate results than Euler–Bernoulli theory, and the number of studies using this theory is limited. Considering the influence of masses on a shaft or beam is very important due to the decrease of natural frequencies of the shaft or beam in the presence of concentrated masses. This reduction should be considered in designing and manufacturing of structures, shafts and other applications.

As mentioned above, some researchers were studied the vibration of the beams with concentrated masses by Euler–Bernoulli theory. Laura et al. [1] studied an Euler–Bernoulli Cantilever beam with a point mass and disregarded the effect

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of shear and rotary inertia. He studied only one boundary condition and considered one position for a concentrated mass. The transverse vibration of a beam with an arbitrary placed concentrated mass and elastically restrained-hinged boundary condition at both ends was conducted by Goel [2]. He used Dirac's delta to impose the effect of one concentrated mass to governing equation and used Laplace transform in his solution. Parnell and Cobble [3] studied lateral displacement of a vibrating Cantilever beam with a concentrated mass with general boundary condition by Laplace transform method. They also considered one position for the point mass. A research on vibration of a Cantilever beam with a concentrated mass and base excitation was carried out by To [4]. He imposed the effect of distance between tip mass center of gravity and point of its attachment to end of the beam.

Grant [5] investigated the influence of rotary inertia and shear deformation or Timoshenko beam theory, on the frequency and normal mode of uniform beams carrying a concentrated mass. He used Dirac's delta function to represent the effects of the concentrated mass on Timoshenko beam and then solved governing equations by Laplace transform method. Bruch and Mitchell [6] studied vibration of a Clamped-Free Timoshenko beam which carries lumped mass-rotary inertia on its free end. He considered the effect of shear force and rotating inertia of lumped mass-rotary inertia and imposed these two effects as a boundary condition at the free end of the beam, then he proved the reduction of first five natural frequencies of beam due to increasing mass or rotating inertia of lumped mass-rotary inertia. Abramovich and Hamburger [7] considered the effect of distance between the tip mass centroid and the point of tip mass attachment on the transverse vibration of a Cantilever beam carrying a tip mass at its free end. This effect causes a moment at the end of beam, and accompanied by effects of shear force and rotating inertia of tip mass. He compared the obtained results with results of Bruch [6]. In another research [8] Abramovich and Hamburger restudied vibration of a uniform Cantilever Timoshenko beam with translational and rotational springs and with a tip mass. In Ref. [9] Rossi et al. investigated free vibrations of Timoshenko beams carrying elastically mounted concentrated masses. He used governing equations of Timoshenko beam, and then compatibility conditions were used to impose the effect of shear force which is caused by mass-spring system on transverse vibration of a Timoshenko beam.

In recent years, scientists tried to solve more complex problems related to the effect of concentrated mass on vibration of beams. Salarieh and Ghrashi [10] studied the effect of finite mass on both torsional and transverse vibration of Timoshenko beam. Free vibration analyses of an immersed beam carrying an eccentric tip mass with rotary inertia is performed by Wu and Hsu [11]. Lin and Tsai [12] used Euler–Bernoulli beam theory to analyze a uniform multi-span beam carrying multiple spring-mass systems. They only used Pinned–Pinned boundary condition for the concerned beam. Finally some researchers used numerical procedures to investigate free vibrations of non-uniform beams carrying concentrated mass or masses. Matsuda et al. [13] presented a method for vibration analysis of tapered Timoshenko beam carrying a tip mass at its end and solved governing differential equations by transforming them into integral equations and integrating them numerically. A DQEM (differential quadrature element method) for vibration of non-uniform Timoshenko beam carrying concentrated masses and rotary inertia with elastic supports was presented by Karami et al. [14]. Transfer matrix method used by Wu and Chen [15] for analyzing free vibration of multi-step Timoshenko beam with eccentric lumped masses and rotary inertias. Free vibration of elastically restrained Cantilever tapered beam carrying concentrated mass and damper was performed by De Rosa et al. [16]. They used symbolic Mathematical software in order to find free vibration frequencies of the Euler–Bernoulli beam. The disadvantages of analytical research which investigated the effect of point masses on the vibration of Timoshenko beam were their limitation on the number of studied point masses, the number of considered boundary conditions, and the position of point masses. Also, they have not presented a general closed form solution for this issue.

In this paper an exact closed form solution for the analysis of the transverse vibration modes of Timoshenko beam carrying multiple concentrated masses is presented. As indicated in results, the number of concentrated masses is not limited, and they can be positioned anywhere on the beam with different boundary conditions.

2. Constitutive correlations

A uniform Timoshenko beam is depicted in Fig. 1. As it can be observed in Fig. 1, only transverse vibration is taken account. In this figure, $y(x,t)$ is transverse displacement of beam, $f(x,t)$ is lateral force imposed to the beam, E is Young's module, I is cross section moment of inertia about beam's neutral axis, ρ and A are density and cross section area respectively, and L is length of the beam. The concentrated mass M_i, which is located at the position x_i, is distributed in dx and is presented as $m_i(x)$, where $M_i = \bar{m}_i \mathrm{d}x$ and it can be modeled by Dirac's delta function as $M_i\delta(x - x_i)$.

Fig. 2 shows the positive sign convention of shear forces and moments in the beam element, where $Q(x,t)$ is shearing force and $M(x,t)$ is bending moment. In Timoshenko beam theory these forces are considered as Eq. (1).

$$\begin{aligned} M(x,t) &= EI\frac{\partial\psi}{\partial x} \\ Q(x,t) &= K_s GA\gamma(x,t) \end{aligned} \tag{1}$$

where ψ is beam rotation, and γ is shear strain. Using the second Newton law and Euler law, the motion equations of beam would be extracted as Eqs. (2) and (3).

$$f(x,t)\mathrm{d}x + Q + \frac{\partial Q}{\partial x}\mathrm{d}x - Q = (\rho A\mathrm{d}x + m_i(x)\mathrm{d}x)\frac{\partial^2 y}{\partial t^2} \tag{2}$$