Applied Mathematics and Computation 238 (2014) 397–407

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the true extrema of Young's modulus in hexagonal materials

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ARTICLE INFO

Keywords:
Classical linear elasticity
Anisotropy
Bounds on effective properties
Historical mechanics of deformable solids: reprintings of classics
Surfaces in Euclidean space

ABSTRACT

In a paper, which has been recently published in *Applied Mathematics and Computations*, Khan and Ahmad (2012) [1] deal with the detection of the extrema of Young's modulus, E, in hexagonal materials.

A few issues presented in that paper, which deserve being outlined and thoroughly discussed, are tackled.

Moreover, in the case of hexagonal materials, a suitable classification is suggested, an exhaustive panoramic view of the possible shape of the surface $E(\mathbf{n})$ generated by Young's modulus for all possible orientations $\mathbf{n}$ is illustrated, and some meaningful numerical examples are proposed.

1. Introduction

A recently published paper [1], by Khan and Ahmad, which will be referred to in the sequel, shortly, as K&A, tackles the problem of finding the extrema of Young's modulus for hexagonal materials.

There are in such a paper a few issues which deserve some discussion, and they will be dealt with in the next Sections. Criticism, of course, is not intended at all in embarrassing the above mentioned authors, but aims at widening the understanding of elastic properties of anisotropic materials belonging to the hexagonal symmetry class.

Specifically the core of the discussion, which is shortly outlined here, is threefold:

1. the bibliography presents inconsistent entries (mismatch between paper title and volume–page numbers) which have been uncritically reproduced by another author: a need to stop this chain, preventing the propagation of incorrect bibliographic data, is here intended, and will be addressed in Section 2;
2. the advantage of using the particular method chosen by the authors (and based on invariants) for seeking the extrema of Young's modulus is questioned, in comparison with traditional techniques based on constrained optimization and will be dealt with in Section 3, where the fundamental properties of the direction surface $E(\mathbf{n})$ will be outlined;
3. a detailed account of the possible occurrence of either two or three extrema of Young's modulus in the upper half of the meridian section of the direction surface $E(\mathbf{n})$ is given in Section 4, along with a thorough discussion of all possible classes of behavior of hexagonal materials, based on two suitably chosen material parameters. The regular missing of one extremum point in the analysis performed by K&A is also shown.

Finally, in Section 5 applications to some real materials belonging to the hexagonal elastic symmetry class are presented and some conclusions are drawn.

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http://dx.doi.org/10.1016/j.amc.2014.04.012
0096-3003/

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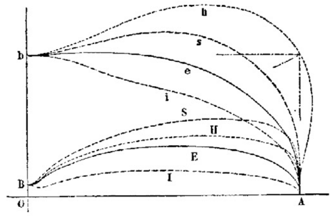
Et les différences avec l'ellipse sont *dans le sens des expériences connues*, car M. Hagen [*], après avoir mesuré les modules d'élasticité de



cinq espèces de bois dans le sens longitudinal et dans le sens transversal supposés être x et y, et avoir trouvé

$$\mathrm{E}_x : \mathrm{E}_y = 48, \quad 83, \quad 15, \quad 22, \quad 23$$

pour les bois de pin, sapin, chêne, hêtre rouge, hêtre blanc,

soit moyennement = 20, a fait aussi quelques mesurages de modules dans des sens obliques x', et a proposé, pour représenter à peu près ses expériences *dont il ne donne pas le détail*, l'expression

$$(155) \qquad \frac{1}{\mathrm{E}} = \frac{c^3_{xx'}}{\mathrm{E}_x} + \frac{c^3_{yx'}}{\mathrm{E}_y},$$

qui fournirait la courbe AHB en petit ponctué; or la courbe pleine AEB donnée par la distribution elliptique des $\sqrt[4]{\mathrm{E}}$, ou par

$$\frac{1}{\mathrm{E}} = \left(\frac{c^2_{xx'}}{\sqrt{\mathrm{E}_x}} + \frac{c^2_{yx'}}{\sqrt{\mathrm{E}_y}}\right)^2$$

n'en diffère pas dans les limites d'exactitude des expériences de ce genre.

La formule empirique de M. Hagen ne saurait être adoptée, car elle donne, quand $\frac{\mathrm{E}_x}{\mathrm{E}_y} = 2$, la courbe Ahb, qui est tout à fait inacceptable; et, en général, comme cette courbe (155) est circonscrite au rectangle

[*] *Ueber die Elasticität des Holzes* (*Annales de Poggendorf*, t. LVIII, p. 125), et *Annales des Ponts et Chaussées*, 1845, 2e semestre, p. 276.

Fig. 1. Reproduction, taken from Gallica-Math (http://portail.mathdoc.fr/JMPA/) of page 424 of de Saint-Venant 1863 paper [3] showing some cross sections of the tension coefficient surface, based on the approximate modeling of experimental data obtained by Hagen [6,7] on wood samples.

2. Historical references

In the paper by K&A [1], reference 1 presents inconsistent entries: title and year do not match volume and page numbers. It is indeed true that Adhémar-Jean-Claude Barré de Saint-Venant (1797–1886) was the first one to study the directional dependence of Young's modulus in anisotropic materials, i.e. of function

$$E = E(\mathbf{n}), \tag{1}$$

$\mathbf{n}$ being the unit vector identifying the direction along which a unit normal (uniaxial) stress state,

$$\boldsymbol{\sigma} = \mathbf{n} \otimes \mathbf{n} \tag{2}$$

is applied.

However his paper *Mémoire sur la distribution des élasticités autour de chaque point d'un solide ou d'un milieu de contexture quelconque, particulièrement lorsqu'il est amorphe sans être isotrope* (i.e. "An essay on the distribution of elastic properties around each point of a solid or of a medium having any texture, particularly when it's amorphous without being isotropic") was published in the *Journal de mathématiques pures et appliquées* (also known as, and with good reasons, the *Journal de Liouville*[1]) in two parts, [2,3], which appeared both in vol. 8 series II (1863), respectively at pages 257–295 and 353–430.

[1] Joseph Liouville (1809–1882) was in 1836 the founder of the *Journal de mathématiques pures et appliquées*, which he served as Editor-in-chief for several years and is still published nowadays. Between 1836 and 1874 he contributed to his *Journal* a large number of contributions under his own name (at least 335 according to the *Catalogue of Scientific Papers 1800–1900* published in 19 volumes by the Royal Society of London—see IV; VIII; X) and under the *nom de plume* of Besge (at least 17, *ibidem*, see I; VII; IX).