Applied Mathematics and Computation 238 (2014) 468–474

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# On cell matrices: A class of Euclidean distance matrices

Pablo Tarazaga [a,*], Hiroshi Kurata [b]

[a] Department of Mathematics and Statistics, Texas A & M University, Corpus Christi, TX 78412, USA
[b] Graduate School of Arts and Sciences, University of Tokyo, Japan

ARTICLE INFO

*Keywords:*
Cell matrices
Euclidean distance matrices
Positive semidefinite matrix

ABSTRACT

In this paper, we study the set of cell matrices and its relationship with the cone of positive semidefinite diagonal matrices. The set forms a convex polyhedral cone in the linear space of symmetric matrices. We describe the faces of the cone and its polar. We also provide a new linear inequality associated with cell matrices.



## 1. Introduction

A Euclidean distance matrix (EDM) is a matrix $D = (d_{ij})$ for which there exist $n$ points $x_1, x_2, \ldots, x_n$ in some Euclidean space $\Re^r$ such that

$$d_{ij} = \|x_i - x_j\|_2^2, \tag{1.1}$$

where $\|\cdot\|_2$ is the usual Euclidean norm. An EDM $D$ is called spherical if the points lie on a sphere in $\Re^r$. (See, for example, [9,1].) During the last two decades, various kinds of subsets of EDMs with particular properties have been studied by several authors. The set of spherical EDMs is a typical example. The set of cell matrices, which is introduced lately by Jaklič and Modic [6] is also an interesting one. As is stated in their paper, the notion of cell matrix is applied to many scientific areas. In this paper, we investigate the structure of the set of cell matrices as a polyhedral cone. We also identify its extreme directions and faces.

For this purpose, let us denote by $S_n$ the linear space of symmetric matrices of order $n$. The Frobenius inner product in $S_n$ will be denoted by $\langle A, B\rangle_F = trace(A^tB)$. The sets of positive semidefinite matrices and hollow symmetric matrices (i.e., symmetric matrices with only zero diagonal entries) are subsets of $S_n$, which will be denoted by $\Omega_n$ and $H_n$, respectively. For a given vector $x = (x_1, x_2, \ldots, x_n)^t \in \Re^n$, $diag(x)$ stands for the diagonal matrix with diagonal entries equal to those of $x$:

$$diag(x) = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}. \tag{1.2}$$

We will denote by $e_i$ $(i = 1, 2, \ldots, n)$ the canonical vectors, and by $e$ the vector of all ones.

The set $\Lambda_n$ of EDMs forms a closed convex cone in $S_n$. The set $\Lambda_n$ is parametrized by the linear transformation $\kappa$ on $\Omega_n$, that is,

* Corresponding author.
*E-mail addresses:* pablo.tarazaga@tamucc.edu (P. Tarazaga), kurata@waka.c.u-tokyo.ac.jp (H. Kurata).

http://dx.doi.org/10.1016/j.amc.2014.04.026
0096-3003/

$$\Lambda_n = \kappa(\Omega_n), \tag{1.3}$$

where $\kappa(B)$ with $B = (b_{ij}) \in \Omega_n$ is defined as,

$$\kappa(B) = be^t + eb^t - 2B \quad \text{with} \quad b = (b_{11}, b_{22}, \ldots, b_{nn})^t \in \Re^n. \tag{1.4}$$

See Gower [4], Critchley [2] and Johnson and Tarazaga [5]. If $\kappa$ is restricted to a maximal face of $\Omega_n$, given by

$$\Omega_n(s) = \{X \in \Omega_n \,|\, Xs = 0\} \quad \text{with} \quad s^t e = 1, \tag{1.5}$$

then the function $\kappa : \Omega_n(s) \to \Lambda_n$ is one to one, and its inverse function $\tau_s : \Lambda_n \to \Omega_n(s)$ is given by

$$\tau_s(D) = -\frac{1}{2}(I - es^t)D(I - se^t). \tag{1.6}$$

Here in general, a face $F$ of a cone $C$ is any subset of $C$ such that for each $a \in F$, every decomposition $a = b + c$ with $b, c \in C$ implies $b, c \in F$. If the dimension $dim(F)$ of $F$ is $dim(C) - 1$, then $F$ is called a maximal face (or a facet). On the other hand, one-dimensional face is called an extreme ray. (See, for example, Chapter 2 of [3].) We call the direction of an extreme ray an extreme direction. Every face $\Omega_n(s)$ with $s^t e = 1$ corresponds to a different location of the origin of coordinates (for more details see Section 2 of [5]). The case in which $s = e/n$ is of particular importance. In this case we write $\tau$ instead of $\tau_{e/n}$. Needless to say, the two functions $\tau : \Lambda_n \to \Omega_n(e)$ and $\kappa : \Omega_n(e) \to \Lambda_n$ are mutually inverse. A matrix in $\Omega_n(e)$ is called centered, since the centroid of the corresponding configuration coincides with the origin.

Given an EDM $D$, we define its embedding dimension as the minimal dimension for which a configuration of the points that generate $D$ can lie. It is well-known that the embedding dimension is the same as the rank of the matrix $\tau_s(D)$ as long as $s^t e = 1$.

Let $\Re^n_+$ be the set of $n$-dimensional vectors whose entries are nonnegative. For $a \in \Re^n_+$, a cell matrix $D(a)$ is defined by

$$(D(a))_{ij} = \begin{cases} 0 & \text{if } j = i, \\ a_i + a_j & \text{if } j \neq i. \end{cases} \tag{1.7}$$

We will denote the set of cell matrices by $\Gamma_n$:

$$\Gamma_n = \{D(a) \,|\, a \in \Re^n_+\}. \tag{1.8}$$

The organization of the paper is as follows. Section 2 gives a characterization of cell matrix via the transformation $\kappa$. It is shown that the set $\Gamma_n$ forms a convex polyhedral cone. Sections 3 and 4 are devoted to describing the faces and the polar cone of $\Gamma_n$, respectively. In Section 5, we derive an interesting implication of cell matrix through a linear inequality.

## 2. Cell matrices structure

In this section we show a natural way to generate the cell matrices, by which a number of interesting properties can be derived. Let $\Delta_n$ be the set of nonnegative diagonal matrices:

$$\Delta_n = \left\{B = diag(b) \,|\, b = (b_1, \ldots, b_n)^t \in \Re^n_+\right\}. \tag{2.1}$$

We begin with the following basic result.

**Theorem 2.1.** *A matrix $D$ is a cell matrix if and only if $D$ can be written as $D = \kappa(B)$ for some $B \in \Delta_n$. That is,*

$$\Gamma_n = \kappa(\Delta_n). \tag{2.2}$$

**Proof.** By the definition of cell matrix, $D \in \Gamma_n$ if and only if there exists a nonnegative vector $a \in \Re^n_+$ such that $D = D(a)$, where the function $D(\cdot)$ is defined in (1.7). Since (1.7) can be rewritten as

$$(D(a))_{ij} = \begin{cases} a_i + a_j - 2a_i & \text{if } j = i, \\ a_i + a_j & \text{if } j \neq i, \end{cases} \tag{2.3}$$

we see that the condition (1.7) is equivalent to $D = \kappa(diag(a))$ for some $a \in \Re^n_+$, which is in turn equivalent to $D = \kappa(B)$ for some $B \in \Delta_n$. This completes the proof. □

By replacing $\Re^n_+$ with $\Re^n$ in the proof of the above theorem, we can see that the above theorem remains valid even if the two sets $\Gamma_n$ and $\Delta_n$ are generalized to $\tilde{\Gamma}_n = \{D(a) \,|\, a \in \Re^n\}$ and $\tilde{\Delta}_n = \{B = diag(b) \,|\, b \in \Re^n\}$, respectively. That is,

$$\tilde{\Gamma}_n = \kappa(\tilde{\Delta}_n). \tag{2.4}$$

We use this fact in the next section.