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Convergence of a general algorithm of asymptotically nonexpansive maps in uniformly convex hyperbolic spaces

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ABSTRACT

In this paper, we establish convergence theorems for a general algorithm of an asymptotically nonexpansive map in a uniformly convex hyperbolic space. Our results generalize simultaneously the approximation results of Rhoades (1994) [18], Suantai (2005) [20] and Xu and Noor (2002) [26] on a nonlinear domain. Our results are refinements and generalizations of the corresponding ones in uniformly convex Banach spaces and $CAT(0)$ spaces.

1. Introduction and preliminaries

Let X be a Banach space. A selfmap T on a nonempty subset C of X is asymptotically nonexpansive [4] if there exists a sequence $\{k_n \geqslant 1\}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n\|x - y\| \quad \text{for all} \quad x, y \in C,\ n \geqslant 1.$$

Noor [17], in 2000, introduced a three-step algorithm and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [3] applied three-step algorithms for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory.

In 2002, Xu and Noor [26] introduced a three-step algorithm to approximate fixed point of asymptotically nonexpansive maps in Banach spaces. Suantai [20] introduced a three-step algorithm in Banach spaces which is simultaneously an extension of the three-step and the two-step algorithms of Xu and Noor [26] and Ishikawa [5].

The modified Xu-Noor algorithm of Suantai [20] is defined as follows.

Let C be a nonempty convex subset of a Banach space X and $T : C \to C$ be a given map. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n)x_n, \quad n \geqslant 1 \end{aligned} \tag{1.1}$$

where $0 \leqslant a_n, b_n, c_n, \alpha_n, \beta_n \leqslant 1$.

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If $c_n = \beta_n = 0$ in (1.1), then it reduces to Xu-Noor algorithm [26]:

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n, \\ y_n &= b_n T^n z_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geqslant 1 \end{aligned} \tag{1.2}$$

where $0 \leqslant a_n, b_n, \alpha_n \leqslant 1$.

For $a_n = c_n = \beta_n = 0$ in (1.1), we get Ishikawa type algorithm [5]

$$\begin{aligned} y_n &= b_n T^n x_n + (1-b_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geqslant 1 \end{aligned} \tag{1.3}$$

where $0 \leqslant \alpha_n, b_n \leqslant 1$.

If $a_n = b_n = c_n = \beta_n = 0$, then (1.1) becomes Mann type algorithm [15]

$$x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n \geqslant 1 \tag{1.4}$$

where $0 \leqslant \alpha_n \leqslant 1$.

In 1970, Takahashi [21] introduced a concept of convexity in a metric space (X, d) as follows.

A map $W : X^2 \times [0,1] \to X$ is a convex structure in X if

$$d(u, W(x,y,\lambda)) \leqslant \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $x, y, u \in X$ and $\lambda \in I = [0,1]$. A nonempty subset C of a convex metric space is convex if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in I$.

Recently, Kohlenbach [11] enriched the concept of convex metric space as "hyperbolic space" by including the following additional conditions in the definition of a convex metric space.

(1) $d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = \mid \lambda_1 - \lambda_2 \mid d(x,y)$
(2) $W(x,y,\lambda) = W(y,x,1-\lambda)$
(3) $d(W(x,z,\lambda), W(y,w,\lambda)) \leqslant \lambda d(x,y) + (1-\lambda)d(z,w)$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$. All normed spaces and their subsets are hyperbolic spaces as well as convex metric spaces. The class of hyperbolic spaces is properly contained in the class of convex metric spaces ([9,11]).

For the definition of a $CAT(0)$ space, basic properties and details to introduce a convex structure in it, we refer to [2,8].

It is remarked that every $CAT(0)$ space is a hyperbolic space [7] (see also [1]).

A hyperbolic space (X, d, W) is uniformly convex [19] if for all $u, x, y \in X, r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $d(W(x,y,\frac{1}{2}), u) \leqslant (1-\delta)r$, whenever $d(x,u) \leqslant r,\ d(y,u) \leqslant r$ and $d(x,y) \geqslant \epsilon r$.

A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for $u, x, y \in X,\ r > 0$ and $\varepsilon \in (0,2]$, is called modulus of uniform convexity of X. We call η to be monotone if it decreases with r (for a fixed ϵ).

A sequence $\{x_n\}$ in a metric space (X, d) has limit existence property for T if $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T) = \{x \in C : Tx = x\}$, the set of fixed points of T, and approximate fixed point property for T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

In 2005, Tian [23] introduced modified convex structure in metric spaces to construct an explicit algorithm of asymptotically quasi-nonexpansive maps with errors as follows:

A map $W : X^3 \times I^3 \to X$ is a modified convex structure on X if for any $(x, y, z; a, b, c) \in X^3 \times I^3$ with $a + b + c = 1$ and $u \in X$, we have

$$d(u, W(x,y,z;\alpha,\beta,\gamma)) \leqslant \alpha d(u,x) + \beta d(u,y) + \gamma d(u,z).$$

Recently, Lee [13], Wang and Liu[24] and Wang et al. [25] established strong convergence theorems for different algorithms in a modified convex metric space. The modified convex structure neither contains Takahashi's convex structure as a special case nor it is suitable to obtain an approximate fixed point sequence in a uniformly convex metric space; in particular, all our results in Section 2 can not be proved in a modified convex metric space due to lack of uniform convex structure therein. Thus our work brodens the scope and unifies approximation fixed point theory in uniformly convex hyperbolic spaces.

Based on some geometrical properties of a convex metric space (not available in a modified convex metric space [22]), we translate the algorithm (1.1) in a convex metric space as follows:

Let C be a nonempty convex subset of a convex metric space X and $T : C \to C$ be a given map. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} z_n &= W(T^n x_n, x_n, a_n), \\ y_n &= W\left(T^n z_n, W\left(T^n x_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(T^n y_n, W\left(T^n z_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \quad n \geqslant 1, \end{aligned} \tag{1.5}$$

where $0 \leqslant a_n, b_n, c_n, \alpha_n, \beta_n, b_n + c_n, \alpha_n + \beta_n \leqslant 1$.