Applied Mathematics and Computation 238 (2014) 245–249

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on the regularity criterion for 3D MHD equations in $\dot{B}^{-1}_{\infty,\infty}$ space

Samia Benbernou [a,*], Mekki Terbeche [b], Maria Alessandra Ragusa [c], Zujin Zhang [d]

[a] Department of Mathematics, University of Mostaganem, Box 227, Mostaganem 27000, Algeria
[b] Department of Mathematics, Faculty of Sciences, University of Oran, Box 1524, Oran 31000, Algeria
[c] Dipartimento di Matematica e Informatica, Universit a di Catania, Viale Andrea Doria, 6, 95125 Catania, Italy
[d] School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China

ARTICLE INFO

Keywords:
Magneto-hydrodynamic equations
BMO space
Weak solution
Regularity criterion

ABSTRACT

In this note, we consider sufficient conditions for the regularity of Leray Hoph Hopf solutions of the 3D incompressible magnetohydrodynamic equations via the velocity and magnetic fields in terms of $\dot{B}^{-1}_{\infty,\infty}$ spaces. We prove that if $(\nabla \times u, \nabla \times B)$ belongs to the space $L^2\left(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right)$, then the solution (u,B) is regular. This extends recent results contained in Gala (2011) [3].

1. Introduction

We are interested in the regularity of weak solutions to the viscous incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + (u.\nabla)u - (B.\nabla)B - \Delta u + \nabla\left(p + \frac{S}{2}|B|^2\right) = 0, \\ \partial_t B + (u.\nabla)B - (B.\nabla)u - \Delta B = 0, \\ \nabla.u = \nabla.B = 0 \end{cases} \tag{1.1}$$

with the following initial conditions

$$\begin{cases} u(x,0) = u_0(x), \\ B(x,0) = B_0(x), \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^3$ and $t \geqslant 0$. Here $u = (u_1,u_2,u_3)$, $B = (B_1,B_2,B_3)$ and $P = p + \frac{1}{2}|B|^2$ are non-dimensional quantities corresponding to the flow velocity, the magnetic field and the total kinetic pressure at the point (x,t), while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla.u_0 = 0$ and $\nabla.B_0 = 0$, respectively. This model describes some important physical phenomena. In particular, for a plasma composed of two types of fluids and formed by ions and electrons, this model can explain the phenomena of fast magnetic reconnection such as in solar flares which cannot be characterized appropriately by the one-fluid magnetohydrodynamics (see e.g. [1,2]).

* Corresponding author.
E-mail addresses: sgsasam73@gmail.com (S. Benbernou), terbeche2000@yahoo.fr (M. Terbeche), maragusa@dmi.unict.it (M.A. Ragusa), zhangzujin361@163.com (Z. Zhang).

http://dx.doi.org/10.1016/j.amc.2014.03.095

It is well-known [9] that the problem (1.1) is locally well-posed for any given initial data $u_0,\ B_0 \in H^s(\mathbb{R}^3),\ s \geqslant 3$, where $H^s(\mathbb{R}^3)$ denotes the usual Sobolev space. But whether this unique local solution can exist globally is an outstanding challenge problem. Some fundamental Serrin's-type regularity criteria in term of the velocity only was done in [5]. Recently, some improvement and extension was made based on these two basic papers. Part of them are listed here: Chen et al. [3] proved regularity by adding condition on the velocity in the Besov spaces; Zhou and Gala [10] proved regularity for u and ∇u in the multiplier space.

For the Navier–Stokes equations, recently, Gala [4] establishes a regularity criterion involving the integrability of $\nabla \times u\ u$ in terms of the Besov spaces $\dot{B}^{-1}_{\infty,\infty}$ and

$$\omega = \nabla \times u \in L^2\left(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right). \tag{1.3}$$

In this paper, inspired by the paper of Gala [4], we prove a regularity criterion $(\nabla \times u, \nabla \times B) \in L^2\left(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)\right)$ for weak solutions to the MHD equations in three-space dimensions. It is a natural way to extend the space widely and improve the previous results.

2. Preliminaries and main result

We begin this section with some notations and lemmas used later. Let $e^{t\Delta}$ denote the heat semi-group defined by

$$e^{t\Delta}f = K_t * f, \quad K_t(x) = (4\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

for $t > 0$ and $x \in \mathbb{R}^3$, where $*$ means convolution of functions defined on $\mathbb{R}^3$.

We now recall the definition of the homogeneous Besov space with negative indices $\dot{B}^{-\alpha}_{\infty,\infty}$ on $\mathbb{R}^3$ with $\alpha > 0$. It is known that $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$ if and only if $e^{t\Delta} \in L^\infty$ for all $t > 0$ and $t^{\frac{\alpha}{2}}\left\|e^{t\Delta}f\right\|_\infty \in L^\infty(0,\infty;L^\infty)$. The norm of $\dot{B}^{-\alpha}_{\infty,\infty}$ is defined, up to equivalence, by

$$\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}} = \sup_{t>0}\left(t^{\frac{\alpha}{2}}\left\|e^{t\Delta}f\right\|_\infty\right).$$

The following lemma is essentially due to Meyer–Gerard–Oru [8], which plays an important role for the proof of our theorem.

Lemma 2.1. *Let $1 < p < q < \infty$ and $s = \alpha\left(\frac{q}{p} - 1\right) > 0$. Then there exists a constant depending only on $\alpha,\ p$ and q such that the estimate*

$$\|f\|_{L^q} \leqslant C\left\|(-\Delta)^{\frac{s}{2}}f\right\|^{\frac{p}{q}}_{L^p}\|f\|^{1-\frac{p}{q}}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{2.1}$$

holds for all $f \in \dot{H}^s_p(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3)$, where $\dot{H}^s_p$ denotes the homogeneous Sobolev space.

In particular, for $s = 1,\ p = 2$ and $q = 4$, we get $\alpha = 1$ and

$$\|f\|_{L^4} \leqslant C\|f\|^{\frac{1}{2}}_{\dot{H}^1}\|f\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}} \tag{2.2}$$

for all $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

Let us recall by Biot–Savart law, for the solenoidal vectors u, the following representation:

$$\frac{\partial u}{\partial x_j} = \mathcal{R}_j(\mathcal{R} \times \omega), \quad j = 1,2,3, \quad \text{where } \omega = \nabla \times u, \tag{2.3}$$

where $\mathcal{R} = (\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)$ and $\mathcal{R}_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. It is known by Jawerth [6] that

$$\|\nabla u\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \leqslant C\|\omega\|_{\dot{B}^{-\alpha}_{\infty,\infty}}. \tag{2.4}$$

Notice that

$$f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^3) \iff \overrightarrow{\nabla}f \in \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

Let

$$C^\infty_{0,\sigma}(\mathbb{R}^3) = \left\{\varphi \in \left(C^\infty_0(\mathbb{R}^3)\right)^3 : \text{div } \varphi = 0\right\} \subseteq \left(C^\infty_0(\mathbb{R}^3)\right)^3.$$

The subspace

$$L^2_\sigma(\mathbb{R}^3) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^3)}^{\|\cdot\|_{L^2}} = \left\{u \in L^2(\mathbb{R}^3)^3 : \text{div } u = 0\right\}$$