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A bilevel programming approach to double optimal stopping

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ABSTRACT

This paper treats a class of double optimal stopping problems arising in the pricing of integral options. Under certain conditions, we give an explicit form of the double stopping time for such type of optimal stopping problems. The present results are essentially derived by solving a certain nonlinear bilevel programming problem explicitly.

1. Introduction

Optimal stopping problems arising in the pricing of integral options have been treated by Kramkov and Mordecki [13] within infinite horizon. Recently, Dai and Zhong [5] have also examined a similar type of optimal stopping problems but in a finite horizon. It must be stressed that the papers [5,13] deal mainly with optimal stopping problems with one stopping time [16]. In this paper, we consider a class of double optimal stopping problems which extends and complements those treated in [5,13]. Under certain conditions, we give an explicit form of the double stopping time for such type of optimal stopping problems. The present results are essentially derived by solving a certain nonlinear bilevel programming problem explicitly. As far as we know, this is the first time the bilevel programming approach is extended to double optimal stopping problems. For an excellent exposition on bilevel programming, see for instance [1,2,4,7,18] and references given therein.

In the next section, we shall now consider a double optimal stopping problem arising naturally in the class of optimal stopping problems treated in [5,13]. The original motivation of the present problem is a class of double optimal stopping problems extensively treated in [11,12,17] and earlier by Haggstrom [9]. However, we note that the present double optimal stopping problem has not been treated elsewhere and our method of approach is also new.

2. Statement of the double optimal stopping problem

Let $Q(t) = (X(t), Y(t))$ be a two-dimensional degenerate process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ starting at (x, y), given by

$$\begin{aligned} dX(t) &= \mu X(t)dt + \beta X(t)dB(t); \quad X(0) = x, \\ dY(t) &= X(t)dt; \quad Y(0) = y, \end{aligned} \tag{2.1}$$

where $\beta > 0,\ \mu \in \mathbf{R}$ are fixed constants and $B(t)$ is a standard one-dimensional Brownian motion.

Let $\mathcal{S}$ denote the set of all stopping times with respect to the filtration $\mathcal{F}^Q$ generated by the process Q, and let $\mathcal{X} = \{(\tau, \sigma) : \tau < \sigma \text{ for } \tau, \sigma \in \mathcal{S}\}$.

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In this paper, we consider the following double optimal stopping problem. Find explicit forms for the stopping times τ^* and σ^*, if they exist, such that

$$\mathbf{E}^{x,y}\left[\frac{X(\sigma^*)}{Y(\tau^*)}\right] = \sup_{(\tau,\sigma)\in\mathcal{X}} \mathbf{E}^{x,y}\left[\frac{X(\sigma)}{Y(\tau)}\right], \tag{2.2}$$

where the supremum is taken over the set of all admissible stopping times $\mathcal{X}$, $\mathbf{E}^{x,y}$ denotes the expectation with respect to the law $\mathbf{P}^{x,y}$ of the process Q starting at (x,y).

It is worth noting that in the particular case of one stopping rule a similar type of optimal stopping problem is treated in [14]. The result therein is essentially derived by reducing the original optimal stopping problem to a problem of solving a multiplicative programming problem. It is shown here that this kind of reduction can also be extended to the double optimal stopping problem (2.2), where a bilevel programming approach (see [1,2,4,7,18] and etc.) now plays an important role. This is the essence of the next section.

3. Explicit solution to a bilevel programming problem

In this section, using the Beibel–Lerche transformation (see [3,14]), we shall consider a certain nonlinear bilevel programming problem (3.2) and (3.3). The resolution of this bilevel programming problem leads to an explicit form of the double optimal stopping time (τ^*,σ^*) for the problem (2.2).

Let $D_0 = \{(x,y)|x > 0,\ y > 0\}$ be a fixed unbounded domain. Consider the following second order partial differential equation

$$\frac{1}{2}\beta^2x^2\Phi_{xx}(x,y) + \mu x\Phi_x(x,y) + x\Phi_y(x,y) = 0 \text{ on } D_1 \tag{3.1}$$

associated with the double optimal stopping problem (2.2), where $D_1 \subset D_0$ and is sought for.

Let $\Phi_k(x,y) \in C^2(D_0)$ $(k = 1,2)$ be strictly positive solutions of Eq. (3.1). Now consider the following nonlinear bilevel programming problem:

$$\min_{x,y} \frac{\frac{x}{y}}{\Phi_1(x,y)}, \tag{3.2}$$

where $\frac{x}{y}$ solves

$$\max_{\frac{x}{y}} \frac{\frac{x}{y}}{\Phi_2(x,y)} \tag{3.3}$$

subject to $(x,y) \in D_0$.

Throughout the paper, we shall let $\Psi_1(x)$ be a positive, concave, strictly increasing function and $\Psi_2(z)$ be a positive, convex, strictly increasing function satisfying

$$\frac{1}{2}\beta^2x^2\Psi_1''(x) + \mu x\Psi_1'(x) = x \tag{3.4}$$

and

$$\frac{1}{2}\beta^2z^2\Psi_2''(z) + (\mu z - z^2)\Psi_2'(z) = 0 \tag{3.5}$$

for all $x,z \in (0,\infty)$.

In the next result, we shall give explicit forms for the bilevel programming problem (3.2) and (3.3) under certain restrictions. The result plays a key role in Section 4 which contains the main result of this paper. The proof of the result follows using the first-order necessary conditions of optimality [15].

Theorem 3.1. *If $\Phi_1(x,y) = \Psi_1(x) - y$ and $\Phi_2(x,y) = \Psi_2\left(z = \frac{x}{y}\right)$ for $x,y \in D_0$, where Ψ_1 and Ψ_2 satisfy (3.4) and (3.5), respectively. Then,*

(a) *$y = x\Psi_1'(x)$ minimizes problem (3.2), provided that $\Psi_1(\theta_k) - 2\theta_k\Psi_1'(\theta_k) = 0$ holds for positive constants θ_1,θ_2 such that $0 < \theta_1 < \theta_2$,*
(b) *$y = \frac{x}{z_*}$ maximizes problem (3.3), where z_* is a unique positive root of the equation*

$$\Psi_2(z_*) - z_*\Psi_2'(z_*) = 0. \tag{3.6}$$

Remark 3.1. It follows from the above result that

$$\min_{x,y} \frac{\frac{x}{y}}{\Phi_1(x,y)} = \frac{1}{\Psi_1(x)\Psi_1'(x) - x[\Psi_1'(x)]^2}$$