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A new projection-based stabilized method for steady convection-dominated convection–diffusion equations

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ABSTRACT

We propose a new projection-based stabilized method for the steady convection-dominated convection–diffusion equations. This method directly increases the L^2-stability instead of the H^1-stability which the variational multiscale (VMS) methods do. Numerical analysis and numerical experiments illustrate and confirm that our new method has effective numerical performances on convection-dominated convection–diffusion problems.

1. Introduction

The standard Galerkin finite element methods (FEM) for the convection-dominated convection–diffusion problems may produce approximate solutions with large nonphysical oscillations unless the space subdivision step is smaller than the diffusion coefficient. In the past two decades, the SUPG (Stream Upwind Petrov Galerkin) methods proposed by Brooks et al. [1] has been a popular choice to solve convection-dominated convection–diffusion problems, since it has good stability without requiring the space subdivision step and high-order accurate compared to the standard Galerkin FEM and straightforward artificial viscosity (AV) FEM. The idea of Petrov Galerkin method also has been used to stabilized pressure, which called PSPG (Pressure stabilized Petrov Galerkin) method. In some work, the SUPG/PSPG have been referred as GLS (Galerkin-Least Square) methods. Lots of works ([2–10], for example) have been devoted to developing these methods.

However, the SUPG/PSPG methods have some undesirable features in application: it introduces additional nonphysical coupling term(s); it produces inaccuracy numerical solutions near the boundary; it has to calculate second derivative when using high order elements.

To overcome those undesirable features of SUPG/PSPG methods, alternative methods such as variational multiscale (VMS) methods [11–18], CIP methods [19], orthogonal subscales methods [17] and local projection stabilized (LPS) methods [20,21] have been proposed and developed in recent years. The VMS method was first proposed by Hughes [11]. The basic idea of VMS methods is to decompose the solution into resolved large scale, resolved fine scale and unresolved scale, then neglect the unresolved scale and define an artificial term acting only on the finest scales, at last add this AV term to the standard Galerkin formulation. It is worth to mention that orthogonal subscales method and LPS method are both special cases of VMS methods, which not only can be used to stabilize the convection-dominated convection–diffusion problems, but also can be used to stabilize pressure when facing mixed problems.

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In this paper, we proposed a new projection-based stabilized method for the convection-dominated convection–diffusion equation. This new method directly increases the L^2-stability instead of the H^1-stability which the VMS methods do. This means that our methods give an extra bounding of the L^2-norm instead of H^1-norm (see Remark 3). Stability and error estimates are proved. The numerical experiments show that our method has quite comparable numerical performances with the SUPG and VMS methods. Moreover, the numerical performances of our method are better than SUPG and VMS methods for some examples. We notice that the similar stabilization term as our method has been used to stabilize the pressure for Stokes equations using low-order mixed finite elements [13]. However, as far as we know, there are no researches on using this type of term to stabilize convection-dominated diffusion problems.

An outline of this paper is as follows. In Section 2, we introduce necessary notations. In Section 3 we propose and analysis our new projection-based method for the convection-dominated convection–diffusion equations. In Section 4, connections with our new method, AV methods, SUPG methods and the VMS methods are presented. In Section 5, numerical experiments have been done to confirm that our new projection-based method has the comparable numerical performance with the VMS methods and SUPG method. In Section 6, we conclude the whole paper.

Throughout this whole paper, we use C to denote a positive constant independent of h and ν, not necessarily the same at each occurrence.

2. Basic notations

Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with polygonal or polyhedral boundary $\Gamma = \partial\Omega$. We use $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ to denote the m order Sobolev spaces on Ω, and use $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$ to denote the norm and semi-norm on these spaces. When $p = 2$, we set $H^m(\Omega) = W^{m,p}(\Omega)$, $H_0^m(\Omega) = W_0^{m,p}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,p}$, $|\cdot|_m = |\cdot|_{m,p}$, and the inner product of $H^m(\Omega)$ denoted by $(\cdot,\cdot)_m$, we also let $(\cdot,\cdot) = (\cdot,\cdot)_0$. Vector analogues of the Sobolve spaces along with vector-valued functions are denoted by upper and lower case bold face font, respectively, e.g., $\boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{L}^2(\Omega)$ and b.

Let $\mathcal{T}_h = \{K\}$ be a regular family of simplex and shape regular. For all $K \in \mathcal{T}_h$, let h_K be the diameter of K and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $P_{k+1}(\mathcal{T}_h)$ represent the $k+1$-order continuous piecewise polynomial on decomposition $\mathcal{T}_h$, $P_k^{dc}(\mathcal{T}_h)$ represent the discontinuous piecewise polynomial on decomposition $\mathcal{T}_h$, $k \geqslant 0$; let $P_1^b(\mathcal{T}_h) = \left\{ v_h \in H^1(\Omega) : v_h|_K \in P_1^b(K), \forall K \in \mathcal{T}_h \right\}$, where $P_1^b(K) = P_1(K) + \prod_{i=1}^d \lambda_i^K P_0(K)$, λ_i^K, $i = 1, 2\ldots, d$ are the $d+1$ barycentric coordinate functions of the element K and $P_k(K)$, $k = 0, 1$ is the kth-order polynomial on K.

The problem studied in this paper is convection-dominated convection–diffusion equation as

$$\begin{cases} -\nu\Delta u + \boldsymbol{b}\cdot\nabla u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma, \end{cases} \tag{2.1}$$

where $u = u(\boldsymbol{x}) \in \mathbb{R}$, $f = f(\boldsymbol{x}) \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^d$, $\nabla\cdot\boldsymbol{b} = 0$ and $\boldsymbol{b} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{W}^{1,\infty}(\Omega)$, ν is the viscosity coefficient and $\nu \ll 1$. For $V = H_0^1(\Omega)$, then a variational formulation of problem (2.1) reads as:

Find $u \in V$ satisfying

$$a(u, v) = (f, v), \quad \forall v \in V, \tag{2.2}$$

where $a(u, v) = \nu(\nabla u, \nabla v) + (\boldsymbol{b}\cdot\nabla u, v) = \nu(\nabla u, \nabla v) + \frac{1}{2}(\boldsymbol{b}\cdot\nabla u, v) - \frac{1}{2}(\boldsymbol{b}\cdot\nabla v, u)$.

Eq. (2.2) is well-posed according to Lax–Milgram Theorem. Let $V_h \subset V$ be a finite element space of continuous, piecewise polynomial functions defined over $\mathcal{T}_h$ satisfying the approximate property: $\forall u \in V$, there exists approximation $I_h u \in V_h$ such that (see [23] for detail):

$$\|u - I_h u\|_0 + h|u - I_h u|_1 \leqslant Ch^{r+1}\|u\|_{r+1}, \quad \forall u \in V \cap H^{r+1}(\Omega). \tag{2.3}$$

A direct discrete of (2.2) is: find $u_h \in V_h$ satisfying

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.4}$$

Eq. (2.4) lacks coercivity when the diffusion coefficient ν is very small. Lots of stabilized methods have been devoted to solving this problem, such as straightforward AV method, SUPG method and VMS methods. We will propose a new stabilized method based on projection in the next section.

3. A new stabilized method based on projection

Let $L_H \in L^2(\Omega)$ $(H \geqslant h)$ be a finite element space of continuous or discontinuous piecewise polynomial functions on decomposition $\mathcal{T}_H$ and $L_H \neq V_h$. We define $\pi_H : L^2(\Omega) \to L_H$ be the L^2-projection, i.e., for $u \in L^2(\Omega)$, find $\pi_H u \in L_H$ such that

$$(u, v_h) = (\pi_H u, v_h), \quad \forall v \in L_H. \tag{3.1}$$

We assume L_H is big enough to satisfy the approximate property of π_h: