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On the solution of characteristic value problems arising in linear stability analysis; semi analytical approach

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ABSTRACT

The linear stability analysis is normally governed by equations that constitute an eigenvalue (characteristic value) problem. In this paper, for the first time, two semi analytical algorithms, (1) Differential Transform Method (DTM) and (2) Adomian Decomposition Method (ADM) are examined for solving a characteristic value problem occurring in linear stability analysis. In this paper, the characteristic value problem of Couette Taylor flow is selected because its simple geometry continues to be a paradigm for theoretical studies of hydrodynamic stability. The results show that DTM handles the solution conveniently and accurately. However, this paper limits the use of ADM for solving characteristic value problems. The results indicate that the present algorithm based on DTM could be used as a promising method for solving characteristic value problems.

1. Introduction

In general, the disturbance differential equations arising in linear stability analysis along with their homogeneous boundary conditions constitute an eigenvalue (characteristic value) problem. For example, Couette Taylor flow is a centrifugally induced hydrodynamic instability that occurs in the flow between coaxial cylinders when the inner cylinder is rotated above a critical speed [1]. Ever since G.I. Taylor's pioneering paper in 1923 [2], Taylor flow has continued to attract researchers with its simple geometry both as an experimental device and as a mathematical model (see e.g. Refs. [2–10]). Therefore, the problem of Couette Taylor stability is known as a paradigm in studies dealing with linear stability analysis. Taylor solved the equations of motion and continuity for Couette flow and then added a small disturbance flow. Based on his prior experimental observations, Taylor assumed axisymmetry, a gap narrow compared to the cylinder radii, negligible gravitational effects, and an axially periodic disturbance flow. He then solved the linearized characteristic equation for the eigenvalue corresponding to the growth rate of the disturbance.

The correct physics of the linear stability problem can be obtained only through the calculation of spatially evolving modes [1]. These modes are given in terms of the wave number and for these situations, where the complex wave number is the eigenvalue. The numerical solution of the characteristic value problems by matrix methods requires an efficient means of discretization such as finite differences approach. Chandrasekhar [3], for example, used Bessel functions expansion and proved that the solution is improved in comparison with Taylor's original theoretical investigation. Using the hyperbolic functions expansion and by splitting the original equation to two sub-equations, Gandhi [4] found that the solution procedure with equal precision could be obtained very simpler than any of the aforesaid ones based on the original equation. Lee

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[6] used variational method to derive a relation between the critical Taylor number and the Reynolds number. Orszagl's pioneering work [11] demonstrated the usefulness and accuracy of Chebyshev spectral methods (see e.g. [11,12]) for solving the linear characteristic value problems. Chebyshev methods naturally cluster collocation points in the vicinity of the boundaries and are more accurate than fourth-order finite differences on a stretched mesh with the same number of grid points. Several techniques to solve the characteristic value problem are discussed by Canuto et al. [13] where the Chebyshev-tau method is favored over the Chebyshev collocation matrix method because of the difficulties in the latter concerning the imposition of the boundary conditions. However, to our best of knowledge, assessment of semi analytical approaches for solving characteristic value problems that occur in linear stability analysis has not been studied so far. In the recent years, Adomian Decomposition Method (ADM) [14–16] and Differential Transform Method (DTM) [17–20] have been proven to be very efficient in obtaining approximate solutions of a wide class of differential equations. In the present study, it is shown that DTM handles the solution of a characteristic value problem of Couette Taylor flow very conveniently and accurately. However, it is demonstrated that ADM is not adequately reliable for solving a characteristic value problem. Owing that the problem of Couette Taylor stability is known as a paradigm in studies dealing with linear stability analysis, the same outcomes can be generalized to other types of characteristic value problems.

2. Mathematical model

The linear stability of Couette Taylor problem is governed by the sixth order equation [4,6]

$$(D^2-\alpha^2)^3 v = -T\alpha^2[1-(1-\mu)y]v \tag{1}$$

with the boundary conditions

$$v = (D^2-\alpha^2)v = D(D^2-\alpha^2)v = 0 \ @ \ y = 0,1 \tag{2}$$

Here, $v = v(y)$ is a function, $D = \partial/\partial y$ is an operator, α and μ are assigned (real) constants and T is the well-known Taylor number [2] (the characteristic value parameter). When $\mu = 1$, the solution of Eq. (1) can, in principle, be achieved very simply. For, in this case, the solution of the equation must clearly be of the form $v = \sum_{i=1}^{6} C_i e^{q_i y}$, where C_i are constants and q_i's, occurring in pairs, are the roots of characteristic equation, $(q^2-\alpha^2)^3 = -T\alpha^2$. The requirement that the solution represented by $v = \sum_{i=1}^{6} C_i e^{q_i y}$ satisfies the six boundary conditions (2) leads to a system of six linear homogeneous equations. The determinant of this system must vanish to obtain the non-trivial solution. The condition that the determinant vanish determines a characteristic equation in terms of the unknowns, α and T. For a given value of α, the critical Taylor number, T_c, can thus be determined. The above solution is merely valid for $\mu = 1$. In general case, two semi-analytical methods, ADM and DTM are assessed for solving the characteristic value problem expressed by Eqs. (1) and (2) and finding the critical Taylor numbers, T_c.

3. Basic concepts of ADM and DTM

The basic definitions and the fundamental theorems of the ADM and DTM plus its applicability for some types of integro-differential equations are given in [17–20]. For the convenience of the reader and without loss of generality, the basic concepts of Adomian Decomposition Method and Differential Transform Method are described as following.

3.1. ADM

Consider an ordinary differential equation (ODE),

$$Lv = f - Nv - Rv \tag{3}$$

where $v = v(y)$ is the dependent variable, y is the independent variable, L is the linear invertible part of the differential operator, N is the nonlinear part and R is the rest of the differential operator that is linear and f indicates the source term. Integrating both sides of Eq. (3) in the interval $[0,y]$ m times, Eq. (3) can be rewritten as,

$$v(y) = v(0) + L^{-1}(f - Nv - Rv) \tag{4}$$

where L is assumed invertible and L^{-1} is an m-fold integral operator. The standard Adomian method [14,15] defines the solution $v(y)$ by the series,

$$v(y) = \sum_{k=0}^{N} v_k(y) \tag{5}$$

where, for larger values of N (the number of truncated terms), faster convergence is expected. The components $v_1, v_2, v_3, \ldots$ can be determined recursively by using the following modified recursive relation [14,15],

$$\begin{aligned} &v_0(0) = v(0) \\ &v_{k+1}(y) = L^{-1}(f - Nv_k(y) - Rv_k(y)) \quad k \geqslant 0 \end{aligned} \tag{6}$$