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Approximation properties of Bernstein–Durrmeyer type operators

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ABSTRACT

This paper deals with the approximation of continuous functions by sequences of some modified Bernstein–Durrmeyer type operators that reproduce certain test functions. The orders of approximation of the new versions turn to be at least as good as the one of the genuine Bernstein–Durrmeyer operators. Moreover, by extrapolating techniques recently applied to the classical Bernstein operators, we present a one-parameter family of modified sequences of operators that reproduce certain polynomials and possess that popular genuine sequence as a limit case. Comparisons and some illustrative graphics are also presented.

1. Introduction and preliminaries

From the classical Bernstein operators, defined for $f \in C[0,1]$ as

$$B_n f(x) = \sum_{k=0}^{n} f(k/n) p_{n,k}(x), \quad x \in [0,1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Durrmeyer [10] introduced the following modification which associates with each function f integrable on the interval $[0,1]$ the polynomial

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0,1]. \tag{1}$$

Many authors have considered further modifications that obey the expression

$$M_{n,\alpha,\beta} f(x) = \sum_{k \in I_n} p_{n,k}(x) f(k/n) + (n-\alpha+1) \sum_{k=\beta}^{n-\alpha+\beta} p_{n,k}(x) \int_0^1 p_{n-\alpha,k-\beta}(t) f(t) dt$$

for certain non negative integers α, β and certain index set $I_n \subset \{0,1,\ldots,n\}$. Thus, for $\alpha = 0$, $\beta = 0$ and $I_n = \emptyset$, the definition reduces to (1), i.e. the so-called *classical Bernstein–Durrmeyer operators*, whose first study in depth was carried out by Derriennic [8].

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Around 1987, Chen [7] and Goodman and Sharma [15] considered the special case $\alpha = 2, \beta = 1$ and $I_n = \{0, n\}$, now usually called the *genuine Bernstein–Durrmeyer operators*, which have been intensively studied in many articles (see for instance [11,28,29,33]; here $p_{0,0}(x) = 1$ and we use the general convention $p_{n,k}(x) = 0$ if $n, k \in \mathbb{N}$ do not satisfy the condition $0 \leqslant k \leqslant n$).

In 1997, Gupta [16] dealt with the case $\alpha = 1, \beta = 0$ and $I_n = \{n\}$, and more recently Gupta and Maheshwari [19] and Abel, Gupta and Mohapatra [1] studied respectively the cases $\alpha = 1, \beta = 1, I_n = \{0\}$ and $\alpha = 0, \beta = 1, I_n = \{0\}$. Further properties of these last two sequences were showed in [20,18] respectively.

As no confusion arises, we shall refer to these sequences by just specifying the values of α and β. Thus, for instance, we denote the classical Bernstein–Durrmeyer operators merely as $M_{n,0,0}$.

Many properties of these five sequences of linear operators are known. They all constitute positive approximation processes for functions $f \in C[0, 1]$ which produce algebraic polynomials, they all reproduce constants, but only the genuine Bernstein–Durrmeyer operators, $M_{n,2,1}$, reproduce linear functions, thus interpolating every function $f \in C[0, 1]$ at the points 0 and 1.

From a more general point of view, one can consider the family of operators given in [20] by

$$M_{n,c}f(x) = \sum_{k=0}^{\infty} a_{n,k}(x; c) \int_0^{\infty} b_{n,k}(t; c) f(t)dt,$$

where

$$a_{n,k}(x; c) = (-1)^k \frac{x^k}{k!} \phi_{n,c}^{(k)}(x), \quad b_{n,k}(t; c) = (-1)^{k+1} \frac{t^k}{k!} \phi_{n,c}^{(k+1)}(t)$$

and

$$\phi_{n,c}(x) = \begin{cases} (1 + cx)^{-n/c}, & x \in [0, \infty), \quad c > 0; \\ e^{-nx}, & x \in [0, \infty), \quad c = 0; \\ (1 - x)^n, & x \in [0, 1], \quad c = -1 \end{cases}$$

(see also [21]), and consider as well the following modification of this family given in [32]:

$$G_{n,c}f(x) = n\sum_{k=1}^{\infty} a_{n,k}(x; c) \int_0^{\infty} a_{n+c,k-1}(t; c) f(t)dt + a_{n,0}(x; c) f(0)$$

(see also [25,31]). The Durrmeyer operators (1) correspond to $M_{n,-1}$, and the operators described above as $M_{n,1,1}$ coincide with $G_{n,-1}$ (see [32, Eq. (34)]).

On the other hand, in recent years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. This line of work originated with a paper by King [23], who introduced a modification of the classical B_n that, on one hand, preserved e_0 and e_2 (here and in the sequel we follow the usual notation $e_i(x) = x^i$), and on the other, presented a degree of approximation at least as good as the one of B_n on the interval $[0, 1/3]$. We could mention a long list of papers dealing with this matter, either using or not the same type of technique introduced by King; see for instance [13,5,26,9,11,14,17,6,4].

In 2010, Gupta and Duman [17], starting from the aforementioned operators $M_{n,1,0}$ and $M_{n,1,1}$, faced to this problem. Specifically, they constructed modifications of the previous ones, say $\widetilde{M}_{n,1,0}$ and $\widetilde{M}_{n,1,1}$, that did preserve linear functions and presented an order of approximation at least as good as the ones of $M_{n,1,0}$ and $M_{n,1,1}$ on the intervals $[0.5, 0.6]$ and $[0.4, 0.5]$ respectively. They also raised the question whether it was possible to consider different modifications that hold fixed e_0 and e_2 instead of e_0 and e_1, and mentioned some instances in this respect.

As for the content of the paper, in the next section, from the considerations above, we show that the modifications presented in [17] can be extrapolated to the classical Bernstein–Durrmeyer operator $M_{n,0,0}$, and with better results. Actually, we shall see that the corresponding operators $\widetilde{M}_{n,0,0}$ approximates at least as good as $M_{n,0,0}$ on the whole interval $[1/3, 2/3]$ where the modification makes sense. Notice that the images of a function by the modified operators $\widetilde{M}_{n,1,0}$ and $\widetilde{M}_{n,1,1}$ presented in [17] are defined only on the intervals $[0.5, 1]$ and $[0, 0.5]$ respectively, and the corresponding improvements of the estimates were achieved only in the respective aforesaid subintervals $[0.5, 0.6]$ and $[0.4, 0.5]$.

In Section 3 we compare the modified operators $\widetilde{M}_{n,0,0}$, $\widetilde{M}_{n,1,0}$ and $\widetilde{M}_{n,1,1}$ with the genuine Bernstein–Durrmeyer operators. This appears to be rather natural as we would be comparing operators that share a basic and important shape preserving property, namely to fix linear functions. In the three cases improvements of the estimates are obtained in the whole interval where the modifications make sense.

Finally, we intend to give an answer to the aforementioned question raised in [17], or better said, we give an answer to a more natural one. To the best of our knowledge, there is no sequence of operators that reproduce e_0 and e_2, constructed with the King technique from another sequence of operators that do not fix e_0 and e_1. Hence, in the last section of this paper, we introduce a modification of the genuine Bernstein–Durrmeyer operators that reproduce e_0 and e_2. Actually, beyond that, we construct a one-parameter family of sequences whose elements reproduce the constants and a certain polynomial of degree 2. We show some convergence and shape properties and state comparisons. The genuine Bernstein–Durrmeyer sequence