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Characterization of compactness for resolvents and its applications ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we derive some characterizations of compactness for resolvents as well as the subordination principle associated with the compactness. As applications, we obtain the existence of fractional evolution equations by using Schauder's fixed point theorem. A simple example about fractional heat equation is also given to illustrate our theory.

1. Introduction

Since the concept of resolvent is introduced by Da Prato and Iannelli in [5,6], the theory of resolvent has received much attention over the past thirty years. Many notions associated with resolvent are developed such as integral resolvent, solution operators, α-resolvent operator functions, (a,k)-regularized resolvent, α-order fractional semigroups and so on. All of these notions play a central role in the study of Volterra equations, especially the fractional differential equations. Concerning the literature we refer the reader to the books [18,19], the recent papers [1–4,7–16] and the references therein.

As we all know, there is a characterization on the compact C_0-semigroups (see [17, Theorem 3.3 of Chapter 2]), which is of great importance in the solvability of semilinear differential equations. That is, C_0-semigroup $T(t)$ is compact for $t > 0$ if and only if $T(t)$ is continuous in the uniform operator topology for $t > 0$ and $(\lambda - A)^{-1}$ is compact for $\lambda \in \rho(A)$. Naturally, we may ask whether or not it is true for the resolvent $S_\alpha(t)$ associated with kernel $a(t) = t^{\alpha-1}/\Gamma(\alpha), 0 < \alpha < 1$. To the best of our knowledge, we have not seen any answer to this question.

The main difficulty of this problem is that there is no the property of semigroups for resolvent. So, it seems to be difficult to prove the continuity of resolvent in the uniform operator topology. Fortunately, however, it is true for the analytic resolvent. That is, we can prove the analytic resolvent $S_\alpha(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, we derive the characterizations of compact analytic resolvent $S_\alpha(t)$, i.e., the analytic resolvent $S_\alpha(t)$ is compact for $t > 0$ if and only if $(\lambda^\alpha - A)^{-1}$ is compact for $\lambda^\alpha \in \rho(A)$. Furthermore, we give a subordination principle associated with the compactness by means of the characterizations. That is, if A generates a compact analytic semigroup, it also generates a compact analytic resolvent $S_\alpha(t)$. Finally, we discuss a class of fractional evolution equations governed by linear operator A generating a compact analytic resolvent and obtain the existence of mild solutions by using Schauder's fixed point theorem.

Recently, we note that Lizama and Poblete [13] develop an interesting functional equation associated with general (a,k)-regularized resolvent families, which can replace the property of semigroups to some extend. So one can expect such a functional equation will be of great importance in the theory of resolvents and applications.

The outline of this paper is as follows. In Section 2, we give some basic definitions and results for resolvent. In Section 3, we derive the characterizations of compact resolvent. As applications, in the last section, we prove the existence of mild solutions for a class of fractional evolution equations and give a simple example on fractional heat equation to illustrate our theory.

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2. Preliminaries

Throughout this paper, let $b > 0$ be fixed, $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}_+$ be the set of positive integers, real numbers and nonnegative real numbers, respectively. We denote by X the Banach space with the norm $\|\cdot\|, C([0,b],X)$ the space of all X-valued continuous functions on $[0,b]$ with the norm $\|u\| = \sup\{\|u(t)\|, t \in [0,T]\}, L^p([0,b],X)$ the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p \, dt)^{1/p}$, where $1 \leqslant p < \infty$. Also, we denote by $\mathcal{L}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators.

In the remainder of this paper, we always suppose that $0 < \alpha < 1$ and A be a closed and densely defined linear operator on X.

Definition 2.1. A family $\{S_\alpha(t)\}_{t \geqslant 0} \subseteq \mathcal{L}(X)$ of bounded linear operators in X is called a resolvent (or solution operator) generating by A if the following conditions are satisfied.

(S1) $S_\alpha(t)$ is strong continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;
(S2) $S_\alpha(t)D(A) \subseteq D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geqslant 0$;
(S3) the resolvent equation holds

$$S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)AS_\alpha(s)x \, ds \quad \text{for all } x \in D(A), \ t \geqslant 0. \tag{2.1}$$

Remark 2.2. Since A is a closed and densely defined operator in X, it is easy to show that (2.1) holds for all $x \in X$ and $t \geqslant 0$ (see [3,19]).

For $\omega, \theta \in \mathbb{R}$, let

$$\sum(\omega,\theta) := \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta\}.$$

Definition 2.3. A resolvent $S_\alpha(t)$ is called analytic, if the function $S_\alpha(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathbb{X})$ admits analytic extension to a sector $\sum(0,\theta_0)$ for some $0 < \theta_0 \leqslant \pi/2$. An analytic solution operator $S_\alpha(t)$ is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is $M_1 = M_1(\omega,\theta)$ such that $\|S(z)\| \leqslant M_1 e^{\omega Rez}$ for $z \in \sum(0,\theta)$, where Rez denotes the real part of z.

Definition 2.4. A resolvent $S_\alpha(t)$ is called exponentially bounded if there exist constants $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S_\alpha(t)\| \leqslant Me^{\omega t}, t \geqslant 0$.

Definition 2.5. A resolvent $S_\alpha(t)$ is called compact for $t > 0$ if for every $t > 0, S_\alpha(t)$ is a compact operator.

3. Characterization of compactness for resolvents

As the importance of pseudoresolvent in the theory of C_0-semigroups, we will use the following concept of α-pseudoresolvent.

Definition 3.1. Let $\alpha > 0$ and D be a subset of the complex plane. A family J of bounded linear operators on X satisfying

$$J(\lambda) - J(\mu) = (\mu^\alpha - \lambda^\alpha)J(\lambda)J(\mu) \quad \text{for} \quad \lambda, \mu \in D$$

is called an α-pseudoresolvent on D.

Clearly, a pseudoresolvent is also a 1-pseudoresolvent. Suppose J is an α-pseudoresolvent on D, it follows from the symmetric property of λ and μ that

$$J(\lambda) - J(\mu) = (\mu^\alpha - \lambda^\alpha)J(\mu)J(\lambda) \quad \text{for} \quad \lambda, \mu \in D.$$

Lemma 3.2. *Let $\alpha > 0$ and A be a linear operator on X such that $(\omega^\alpha, \infty) \subset \rho(A), \omega \geqslant 0$. Suppose that $R_\alpha : (\omega,\infty) \to \mathcal{L}(X)$ be defined by $R_\alpha(\lambda) = (\lambda^\alpha - A)^{-1}$ for $\lambda \in (\omega,\infty)$. Then R_α is an α-pseudoresolvent on (ω,∞).*

Proof. Let R be defined by $R(\lambda) = R_\alpha(\lambda^{1/\alpha})$ for $\lambda \in (\omega^\alpha, \infty)$. Then $R(\lambda)$ is a pseudoresolvent on (ω^α, ∞). Thus