Applied Mathematics and Computation 232 (2014) 68–75

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some inequalities for differentiable convex mapping with application to weighted midpoint formula and higher moments of random variables

Dah-Yan Hwang

Department of Information and Management, Taipei Chengshih University of Science and Technology, No. 2, Xueyuan Rd., Beitou, 112 Taipei, Taiwan

ARTICLE INFO

Keywords:
Hermite–Hadamard inequality
Convex function
Wright-convex function
Quasi-convex function
Integral inequalities
Midpoint formula

ABSTRACT

Connected with the celebrated Hermite–Hadamard integral inequality, several new inequalities for differentiable convex, wright-convex and quasi-convex mapping are established. Applications of these results are considered in error estimates for weighted Midpoint integral formula and in deriving the inequalities involving higher moments of random variables.

1. Introduction

Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers $a, b \in I$ and with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This inequality is known as the Hermite–Hadamard inequality for convex functions. In recent years, many authors established several inequalities connected to Hadamard's inequality. For recent results, as refinements counterparts, generalizations and new Hadamard's type inequalities, see [1–9,11].

An important question is estimating the difference between the middle and leftmost terms in (1.1), see [6,7,9,11]. The most representative articles are following Theorem 1.1 given by Kirmaci in [6] and Theorem 1.2 obtained by Pearce and Pečarić in [9]. The main inequalities were pointed out as follows:

Theorem 1.1. *Let $f : I \subseteq R \to R$ be differentiable mapping on I^0, where $a, b \in I$ with $a < b$, and $f' \in L(a,b)$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leqslant \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{1.2}$$

Theorem 1.2. *Let $f : I \subseteq R \to R$ be differentiable mapping on I^0, where $a, b \in I^0$ with $a < b, f' \in L(a,b)$, and $q \geqslant 1$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leqslant \frac{(b-a)}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{1.3}$$

E-mail address: dyhuang@tpcu.edu.tw

http://dx.doi.org/10.1016/j.amc.2014.01.050
0096-3003/

The first, we recall the definition of wright-convex function. In [10, p.7], the mapping $f:[a,b]\to R$ is a wright-convex function on $[a,b]$, if

$$f(x+z)+f(y)\ \leqslant\ f(y+z)+f(x)$$

for each $x\ \leqslant\ y, z\geqslant 0$, and $x, y+z\in[a,b]$ as well as a convex function must be a wright-convex function but not conversely. The second, we recall the definition of a quasi-convex function. In [10, p.7], the mapping $f:[a,b]\to R$ is a quasi-convex function on $[a,b]$, if

$$f(x+(1-)y)\ \leqslant\ \max\{f(x),f(y)\}$$

for all $x,y\in[a,b]$ and $\in[0,1]$. It is also pointed that a convex function must be a quasi-convex function but not conversely.

The main purpose of this paper is to establish some new weighted inequalities of (1.2) and (1.3) involving the class of functions whose derivatives in absolute value at certain powers are convex, and two parallel developments are made based on wright-convexity and quasi-convexity respectively. Applying the obtained results, some estimates of the error term for weighted Midpoint formula are obtained in Section 3, and the r-th moment of continuous random variables having the continuous probability density function are given in Section 4.

In the following sections, for convenient, let the notation $L(a,b,t)=\frac{1+t}{2}a+\frac{1-t}{2}b$ and $U(a,b,t)=\frac{1-t}{2}a+\frac{1+t}{2}b$.

2. Main Results

The following Lemma is necessary and plays important role in this article.

Lemma 2.1. *Let $f:I\subseteq R\to R$ be differentiable mapping on I^0, where $a,b\in I$ with $a<b$, and $h:[a,b]\to[0,\infty)$ be differentiable mapping. If $f'\in L(a,b)$ then the following inequality holds:*

$$\frac{h(a)}{2}(f(a)+f(b))-h(b)f\left(\frac{a+b}{2}\right)+\frac{b-a}{4}\int_0^1\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.$$
$$\left.+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt=\frac{(b-a)}{4}\left\{\int_0^1\left[h\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)-h\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(b)\right]\times\left[-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.\left.+f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}.$$

Proof. By the integration by parts, we have

$$I_1=-\int_0^1\left[h\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)-h\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(b)\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt=\frac{2}{b-a}\left\{\left[h\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.-h\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(b)\right]f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Big|_0^1-\frac{(a-b)}{2}\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.$$
$$\left.\left.+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}=\frac{2}{b-a}\left\{\left[h(a)f(a)-h(b)f\left(\frac{a+b}{2}\right)\right]+\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.\left.+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}$$

and

$$I_2=\int_0^1\left[h\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)-h\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(b)\right]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt=\frac{2}{b-a}\left\{\left[h\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.-h\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(b)\right]f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1-\frac{(a-b)}{2}\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.$$
$$\left.\left.+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}=\frac{2}{b-a}\left\{\left[h(a)f(b)-h(b)f\left(\frac{a+b}{2}\right)\right]+\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.\left.+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}.$$

Therefore,

$$\frac{b-a}{4}[I_1+I_2]=\frac{h(a)}{2}[f(a)+f(b)]-h(b)f\left(\frac{a+b}{2}\right)+\frac{(b-a)}{4}\left\{\int_0^1\left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\right.$$
$$\left.\left.+f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]\times\left[h'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+h'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right\}.$$

This completes the proofs. □