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Positive periodic solutions of second-order differential equations with weak singularities

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ABSTRACT

We establish the existence of positive periodic solutions of the second-order differential equation

$$u'' + a(t)u = f(t,u) + c(t)$$

via Schauder's fixed point theorem, where $a \in L^1(\mathbb{R}/T\mathbb{Z};\mathbb{R}_+),\ c \in L^1(\mathbb{R}/T\mathbb{Z};\mathbb{R}),\ f$ is a Carathéodory function and is singular at $u = 0$. Our main results generalize some recent results by P.J. Torres.

1. Introduction and the main results

In this paper, we are concerned with the existence of positive periodic solutions of the second-order differential equation

$$u'' + a(t)u = f(t,u) + c(t), \tag{1.1}$$

where $a \in L^1(\mathbb{R}/T\mathbb{Z};\mathbb{R}_+),\ c \in L^1(\mathbb{R}/T\mathbb{Z};\mathbb{R}),\ f \in \mathrm{Car}(\mathbb{R}/T\mathbb{Z} \times (0,\infty),\ \mathbb{R})$, which means $f|_{[0,T]} : [0,T] \times (0,\infty) \to \mathbb{R}$ is a L^1-Carathéodory function, and f is singular at $u = 0$.

In the case that $a(t) \equiv 0$ and $f(t,u) = \frac{1}{u^\lambda}$, (1.1) reduces to the special equation

$$u'' = \frac{1}{u^\lambda} + c(t), \tag{1.2}$$

which was initially studied by Lazer and Solimini [1]. They proved that for $\lambda \geqslant 1$ (called *strong force condition* in a terminology first introduced by Gordon [2,3]), a necessary and sufficient condition for the existence of a positive periodic solution of (1.2) is that the mean value of c is negative,

$$\bar{c} := \frac{1}{T}\int_0^T c(t)dt < 0.$$

Moreover, if $0 < \lambda < 1$ (*weak force condition*) they found examples of functions c with negative mean values and such that periodic solutions do not exist.

If compared with the literature available for strong singularities, see [4–15] and the references therein, the study of the existence of periodic solutions under the presence of a weak singularity is much more recent and the number of references is considerably smaller. The likely reason may be that with a weak singularity, the energy near the origin becomes finite, and this fact is not helpful for obtaining a priori bound needed for a classical application of the degree theory, and also is not

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helpful for the fast rotation needed in recent versions of the Poincaré-Birkhoff theorem. The first existence result with weak force condition appears in Rachunková et al. [16]. Since then, Eq. (1.1) with f has weak singularities has been studied by several authors, see Torres [17,18], Franco and Webb [19], Chu and Li [20].

Recently, Torres [18] showed how a weak singularity can play an important role if Schauder's fixed point theorem is chosen in the proof of the existence of positive periodic solution for (1.1). From now on, for a given function $\xi \in L^\infty[0,T]$, we denote the essential supremum and infimum of ξ by ξ^* and ξ_*, respectively. We write $\xi \succ 0$ if $\xi \geqslant 0$ for a.e. $t \in [0,T]$ and it is positive in a set of positive measure. Under the assumption.

(H1) The linear equation $u'' + a(t)u = 0$ is nonresonant and the corresponding Green's function

$$G(t,s) \geqslant 0, \quad (t,s) \in [0,T] \times [0,T].$$

Torres showed the following three results.

Theorem A [18, Theorem 1]. *Let (H1) hold and define*

$$\gamma(t) = \int_0^T G(t,s)c(s)\,ds. \tag{1.3}$$

Assume that

(H2) *there exist $b \in L^1(0,T)$ with $b \succ 0$ and $\lambda > 0$ such that*

$$0 \leqslant f(t,u) \leqslant \frac{b(t)}{u^\lambda}, \quad \text{for all } u > 0, \quad a.e.\ t \in [0,T].$$

If $\gamma_* > 0$, then there exists a positive T-periodic solution of (1.1).

Theorem B [18, Theorem 2]. *Let (H1) hold. Assume that*

(H3) *there exist two functions $b,\ \hat{b} \in L^1(0,T)$ with $b,\ \hat{b} \succ 0$ and a constant $\lambda \in (0,1)$ such that*

$$0 \leqslant \frac{\hat{b}(t)}{u^\lambda} \leqslant f(t,u) \leqslant \frac{b(t)}{u^\lambda}, \quad u \in (0,\infty), \quad \text{a.e. } t \in [0,T].$$

If $\gamma_* = 0$. Then (1.1) has a positive T-periodic solution.

Theorem C [18, Theorem 4]. *Let (H1) and (H3) hold. Let*

$$\hat{\beta}_* = \min_{t\in[0,T]}\left(\int_0^T G(t,s)\hat{b}(s)\,ds\right), \quad \beta^* = \max_{t\in[0,T]}\left(\int_0^T G(t,s)b(s)\,ds\right).$$

If $\gamma^* \leqslant 0$ and

$$\gamma_* \geqslant \left(\frac{\hat{\beta}_*}{(\beta^*)^\lambda}\lambda^2\right)^{\frac{1}{1-\lambda^2}}\left(1 - \frac{1}{\lambda^2}\right).$$

Then (1.1) has a positive T-periodic solution.

Obviously, (H2) and (H3) are too restrictive that the above mentioned results are only applicable to (1.1) with nonlinearity which is bounded in origin and infinity by a function of the form $\frac{1}{u^\lambda}$.

Of course the natural question is what would happen if we allow that the nonlinearity f is bounded by two different functions $\frac{1}{u^\alpha}$ and $\frac{1}{u^\beta}$?

It is the purpose of this paper to study the existence of positive periodic solutions of (1.1) under the more general assumptions.

(A1) $f|_{[0,T]} : [0,T] \times (0,\infty) \to \mathbb{R}$ is a L^1-Carathéodory function.
(A2) There exist $b, e \in L^1(0,T)$ with $b, e \succ 0, \alpha, \beta \in (0,\infty), m \leqslant 1 \leqslant M$, such that

$$0 \leqslant f(t,u) \leqslant \frac{b(t)}{u^\alpha}, \quad u \in (M,\infty), \quad \text{a.e. } t \in [0,T],$$

and

$$0 \leqslant f(t,u) \leqslant \frac{e(t)}{u^\beta}, \quad u \in (0,m), \quad \text{a.e. } t \in [0,T].$$