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A simple approach to the particular solution of constant coefficient ordinary differential equations

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ABSTRACT

An eigenfunction approach to compute the particular solution of constant coefficient ordinary differential equations is presented. It is shown that the exponentials are the eigenfunctions of such equations. Solutions corresponding to products of powers and exponentials are obtained. The singular case is studied and a fast algorithm for its implementation is presented.

1. Introduction

The constant coefficient ordinary differential equations are ubiquitous. Their mathematical manageability and practical usefulness make them odd tools, even if they are not the best, for studying or modeling most natural or artificial systems. This motivates their development since a long time ago and made them object of many books.

Perhaps we should not expect to find new features on such equations. This is a wrong thought as shown in [2,3] and motivates us to propose a different approach for computing the particular solution for the special and of prime importance that is the exponential case.

In engineering applications the exponentials or particularly the sinusoids are used to study the so-called steady state of the filters, name usually given to the systems described by ordinary differential equations. Normally they are written in the general format [5]

$$\sum_{k=0}^{N} a_k D^{\alpha_k} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k} x(t), \tag{1}$$

with $t \in \mathbb{R}$; the symbol D represents the derivative operator. The parameters α_k and β_k are the derivative orders. In the so-called commensurate case we write $\alpha_k = \beta_k = k\alpha$. For now we will consider α equal to 1. The generalisation for fractional orders will be presented in a future paper. On writing the second member with this format it is intended to consider practical implementations where we need to consider derivatives on the input and, if studying the transient behaviour, the corresponding initial conditions. As said above we will treat the steady state solution.

Particularising (1) for $\alpha = 1$ we have

$$\sum_{k=0}^{N} a_k D^k y(t) = \sum_{k=0}^{M} b_k D^k x(t). \tag{2}$$

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In current applications we assume that $M \leqslant N$ for stability reasons. From a pure mathematical point of view, it is not necessary to have such assumption provided that we work in the context of the generalised functions [10]. For wide classes of functions we can use the Laplace transform (two-sided) or the Fourier transform [7] to find the particular solution of (2). However the functions $x(t)$ with the format

$$x(t) = e^{\beta t}\sum_{k=0}^{K} d_k t^k,$$

used in [3] do not have neither Laplace transform nor Fourier transform, even in distributional sense. Jia & Sogabe treated this subject by presenting an explicit formulation for the particular solution of equations of the type stated above, but they put $M = 0$. Other interesting approaches can be found in [1,2,4,8], but we think it is possible to get simpler algorithms. In the following and without loosing generality, because Eq. (2) is linear, we will consider the case

$$x(t) = e^{\beta t} t^K. \tag{3}$$

The paper outlines as follows. We will start by introducing the eigenfunctions of differential equations and compute the corresponding eigenvalues. These are used to calculate the particular solutions we are looking for, namely in the singular case. Several examples are presented to illustrate the behaviour of the approach. Some procedures to make easier the algorithm are described.

2. The eigenfunctions of differential equations

2.1. The exponentials as eigenfunctions

Let $x(t)$ and $y(t)$ be two functions defined on $\mathbb{R}$. The convolution is a binary operation defined with generality by:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\mathrm{d}\tau. \tag{4}$$

As it is easy to verify, the neutral element of the convolution is the Dirac impulse distribution, $\delta(t)$. Following a current procedure in Signal Processing, consider the solution, $y(t)$, of (2) when $x(t)$ is an impulse, $x(t) = \delta(t)$. We call it Impulse Response and represent it by $h(t)$. We have then

$$\sum_{k=0}^{N} a_k D^k h(t) = \sum_{k=0}^{M} b_k D^k \delta(t). \tag{5}$$

Now convolve both sides in (5) with $x(t)$. As known:

$$x(t) = \delta(t) * x(t) \text{ and } \left[D^k h(t)\right] * x(t) = D^k[h(t) * x(t)],$$

This means that the solution of (2) is the convolution of $x(t)$ with the impulse response

$$y(t) = h(t) * x(t) \tag{6}$$

Theorem 1. *Eigenfunction*
The particular solution of the differential Eq. (2) when $x(t) = e^{st}$ $s \in \mathbb{C}$ is given by

$$y(t) = H(s)e^{st}, \tag{7}$$

provided that $H(s)$ exists.

Proof. Insert $x(t) = e^{st}$ into (4) to get

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau e^{st},$$

with

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau, \tag{8}$$

we obtain (7). □

This theorem shows that the exponential defined on $\mathbb{R}$ is the eigenfunction of the constant coefficient ordinary differential Eq. (2). The eigenvalue, $H(s)$, is the *Transfer Function* of the system defined by the differential Eq. (2) and is the Laplace transform of the impulse response.