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Homogenization of a variational problem in three-dimension space ☆

Cuiping Guo [a,*], Weiyu Liu [b]

[a] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
[b] Micro- and Nano-manufacturing Research Center, State Key Laboratory for Manufacturing Systems Engineering, Xi'an Jiaotong University, Xi'an 710049, China

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ABSTRACT

In this paper, we investigate the variational problem for a sequence of 3-dimensional domains with highly oscillating boundaries. Using the unfolding method and the averaging method, we obtain the result of the homogenization problem, that is, a sequence of solutions of Eq. (3.1) converges to the solution of Eq. (3.4) as the periodic length approaches zero. It is noteworthy that the convergence is in the strong sense.

1. Introduction

The periodic unfolding method was introduced in [6] by Cioranescu et al. for the study of classical periodic homogenization in the case of fixed domains and further described in [1–4,8,9,11]. This method was also applied to problems with holes and truss-like structures or in linearized elasticity.

The homogenization of periodic structures was carried out in the last 30 years for various kinds of problems involving differential equations[12–15] and systems, as well as integral energies. But most of these works all got the weak convergence. Recently, there was a break in [5,7], where the achievement of strong convergence was obtained. In [10], the unfolding method was applied to a linear elliptic equation in the oscillating boundary cases in two-dimension space, and the new result of strong convergence was obtained. The purpose of this paper is to generalize the work in [10], i.e. we apply the periodic unfolding method to a variational problem in the oscillating boundary cases in three-dimension space, and obtain the strong convergence result. The symbols used in this paper are the same as the ones to those in [10].

We will work on domains which are constructed as follows. Let $\beta \in \mathbb{R}$ such that $\beta \in (0,1), 1/\varepsilon = N$, where N is a positive integer. Define

$$\Omega_A^\varepsilon = \bigcup_{k=0}^{N-1}(k\varepsilon, k\varepsilon+\beta\varepsilon) \times \bigcup_{l=0}^{N-1}(l\varepsilon, l\varepsilon+\beta\varepsilon) \times (0,1),$$

$\Omega_B = (0,1)\times(0,1)\times(-1,0)$, $\Omega_A^\varepsilon \cap \Omega_B = \Gamma_\varepsilon$, $\Omega_\varepsilon = \Omega_A^\varepsilon \cup \Omega_B \cup \Gamma_\varepsilon$, $\Omega_A = (0,1)^3$, $\Omega = (0,1)\times(0,1)\times(-1,1)$.

2. The unfolding operator

A linear operator on $L^1(\Omega_A^\varepsilon)$ will be defined and used to interpret integrals over ε-dependent domains as integrals over a fixed domain. This operator is called the unfolding operator.

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* Corresponding author.
E-mail address: gcp5056@yahoo.com.cn (C. Guo).

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We will use the following notations: Let $[\cdot]:\mathbb{R}\to\mathbb{Z}$ and $\{\cdot\}:\mathbb{R}\to[0,1)$ denote the functions which map every real number to its integer part and the fractional part.

Definition 2.1 (*The unfolding operator*). For every $\varepsilon>0$, $u\in L^1(\Omega_A^\varepsilon)$, we define the unfolding operator $T^\varepsilon : L^1(\Omega_A^\varepsilon)\to L^1(\Omega_A\times(0,\beta)^2)$ by setting

$$T^\varepsilon(u)(x_1,x_2,x_3,x_4,x_5)=u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)$$

for every $(x_1,x_2,x_3)\in\Omega_A$ and $(x_4,x_5)\in(0,\beta)^2$.

If U is an open subset of $\mathbb{R}^3$ containing Ω_A^ε and u is a real-valued function on U, $T^\varepsilon u$ will mean T^ε acting on the restriction of u to Ω_A^ε. The following propositions state the properties of T^ε which will be used later. Most of them are straightforward and their proofs are omitted.

Proposition 2.1. *T^ε is linear.*

Proposition 2.2. *Let u, v be functions: $\Omega_A^\varepsilon\to\mathbb{R}$, then $T^\varepsilon(uv)=T^\varepsilon uT^\varepsilon v$.*

Proposition 2.3. *Let $x\in\Omega_A^\varepsilon$ and $u:\Omega_A^\varepsilon\to\mathbb{R}$, then $T^\varepsilon u(x_1,x_2,x_3,\{\frac{x_1}{\varepsilon}\},\{\frac{x_2}{\varepsilon}\})=u(x_1,x_2,x_3)=u(x)$.*

Proposition 2.4. *Let $u\in L^1(\Omega_A^\varepsilon)$, then*

$$\int_{\Omega_A\times(0,\beta)^2}T^\varepsilon u dx=\int_{\Omega_A^\varepsilon}udx.$$

Proof. Suppose that $u\in L^1(\Omega_A^\varepsilon)$. By Fubini's theorem and the fact that $T^\varepsilon u$ is piecewise constant in x_1 and x_2,

$$\begin{aligned}\int_{\Omega_A\times(0,\beta)^2}T^\varepsilon u dx&=\int_{x_1=0}^{1}\int_{x_2=0}^{1}\int_{x_3=0}^{1}\int_{x_4=0}^{\beta}\int_{x_5=0}^{\beta}u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)dx_1dx_2dx_3dx_4dx_5\\&=\int_{x_3=0}^{1}\int_{x_4=0}^{\beta}\int_{x_5=0}^{\beta}\sum_{k=0}^{N-1}\int_{x_1=k\varepsilon}^{k\varepsilon+\varepsilon}\sum_{l=0}^{N-1}\int_{x_2=l\varepsilon}^{l\varepsilon+\varepsilon}u(\varepsilon k+\varepsilon x_4,\varepsilon l+\varepsilon x_5,x_3)dx_1dx_2dx_3dx_4dx_5\\&=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\int_{x_3=0}^{1}\int_{x_1=k\varepsilon}^{k\varepsilon+\beta\varepsilon}\int_{x_2=l\varepsilon}^{l\varepsilon+\beta\varepsilon}u(x_1,x_2,x_3)dx_1dx_2dx_3=\int_{\Omega_A^\varepsilon}u(x)dx.\qquad\square\end{aligned}$$

Proposition 2.5. *Let $u\in L^2(\Omega_A^\varepsilon)$, then $T^\varepsilon u\in L^2(\Omega_A\times(0,\beta)^2)$. Moreover, $T^\varepsilon u$ is a linear isometry between $L^2(\Omega_A^\varepsilon)$ and $L^2(\Omega_A\times(0,\beta)^2)$.*

Proof. Suppose that $u\in L^2(\Omega_A^\varepsilon)$, then $|u|^2\in L^1(\Omega_A^\varepsilon)$. By Proposition 2.2 and 2.4, we have:

$$\int_{\Omega_A\times(0,\beta)^2}|T^\varepsilon u|^2dx=\int_{\Omega_A\times(0,\beta)^2}T^\varepsilon|u|^2dx=\int_{\Omega_A^\varepsilon}|u|^2dx<\infty.$$

By the previous calculation we can see that T^ε is a mapping preserving norm, that is $\|T^\varepsilon u\|_{L^2(\Omega_A\times(0,\beta)^2)}=\|u\|_{L^2(\Omega_A^\varepsilon)}$. This Proposition and Proposition 2.1 imply that T^ε is a linear isometry between $L^2(\Omega_A^\varepsilon)$ and $L^2(\Omega_A\times(0,\beta)^2)$. $\square$

Proposition 2.6. *Let $u\in H^1(\Omega_A^\varepsilon)$, then $T^\varepsilon u\in L^2(\underset{x_1}{(0,1)}\times\underset{x_2}{(0,1)};H^1(\underset{x_3}{(0,1)}\times\underset{x_4}{(0,\beta)}\times\underset{x_5}{(0,\beta)}))$. Furthermore, $\frac{\partial}{\partial x_3}T^\varepsilon u=T^\varepsilon\frac{\partial u}{\partial x_3}$, $\frac{\partial}{\partial x_4}T^\varepsilon u=\varepsilon T^\varepsilon\frac{\partial u}{\partial x_1}$ and $\frac{\partial}{\partial x_5}T^\varepsilon u=\varepsilon T^\varepsilon\frac{\partial u}{\partial x_2}$.*

Proof. According to the chain rule, we can obtain

$$\begin{aligned}\frac{\partial}{\partial x_3}T^\varepsilon u&=\frac{\partial}{\partial x_3}u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=u_3\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=T^\varepsilon\frac{\partial u}{\partial x_3},\\\frac{\partial}{\partial x_4}T^\varepsilon u&=\frac{\partial}{\partial x_4}u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=\varepsilon u_1\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=\varepsilon T^\varepsilon\frac{\partial u}{\partial x_1},\\\frac{\partial}{\partial x_5}T^\varepsilon u&=\frac{\partial}{\partial x_5}u\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=\varepsilon u_2\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon x_4,\varepsilon\Big[\frac{x_2}{\varepsilon}\Big]+\varepsilon x_5,x_3\Big)=\varepsilon T^\varepsilon\frac{\partial u}{\partial x_2}.\end{aligned}$$