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Large and entire large solutions for a class of nonlinear problems

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ABSTRACT

In this paper, our main purpose is to establish the existence of entire large positive solutions to the quasilinear equation $\Delta_p u = \beta(x)h(u)$. We give also necessary conditions for the existence and nonexistence of such solutions. We distinguish the cases when β is radial and when it is nonradial.

1. Introduction

In this work, we consider the boundary blow-up elliptic problem

$$\Delta_p u = \beta(x)h(u) \quad \text{in} \quad \Omega, \tag{1.1}$$
$$u = \infty \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is either a bounded domain or the whole space $\mathbb{R}^N$. $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u),\ 2 \leqslant p \leqslant N$, is the well known p-Laplacian operator. We assume throughout this paper that β is a nonnegative locally bounded function on $\mathbb{R}^N$. The nonlinearity h is assumed to fulfill:

(**H1**) h is continuous and nondecreasing on $[0,\infty)$, satisfies $h(0) = 0,\ h(s) > 0$ for $s > 0$.

Such a solution is called a large solution to the elliptic Eq. (1.1) in Ω. In the case $\Omega = \mathbb{R}^N$, a large solution is called an entire large solution. For a bounded domain Ω, the boundary condition (1.2) means that $\lim_{x\to z} u(x) = \infty$ for every $z \in \partial\Omega$.

If $\Omega = \mathbb{R}^N$, we require moreover that $\lim_{|x|\to\infty} u(x) = \infty$.

This problem appears in the study of non-Newtonian fluids [2,16] and non-Newtonian filtration [9,10]. Such problems also arise in the study of the subsonic motion of a gas [17], the electric potential in some bodies [13] and Riemannian geometry [6].

In the situation $p = 2$ and $h(u) = u^\gamma,\ 0 < \gamma < 1$, boundary blow-up problems of the kind (1.1), (1.2) have received much attention during the last years (see, e.g., El Mabrouk and Hansen [8], Lair [11], Lair and Wood [12] and the references therein). For radial locally Hölder continuous function β on $\mathbb{R}^N (N \geq 3)$, Lair and Wood [12] proved that (1.1), (1.2) possesses an entire large solution (which is radial) if and only if

$$\int_0^\infty r\beta(r)dr = \infty.$$

For continuous functions β which are not necessarily radial on $\mathbb{R}^N (N \geq 3)$, Lair [11] assumed that

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$$\int_0^\infty r\beta_{osc}(r)\exp(\psi(r))dr < \infty, \tag{1.3}$$

where

$$\psi(r) = \int_0^r t\inf_{|x|=t}\beta(x)dt \quad \text{and} \quad \beta_{osc}(r) = \sup_{|x|=r}\beta(x) - \inf_{|x|=r}\beta(x).$$

The author proved that (1.1) has an entire large solution if and only if

$$\int_0^\infty r\inf_{|x|=r}\beta(x)dr = \infty. \tag{1.4}$$

Recently, El Mabrouk and Hansen [8] have considered problems of the type (1.1), (1.2) for locally bounded functions β which are not necessarily radial on $\mathbb{R}^N$ for $N \geq 3$. Assuming that the oscillation function $\beta(r)$ is small enough in the sense that

$$\int_0^\infty r\beta_{osc}(r)(1+\psi(r))^{\frac{\gamma}{1-\gamma}}dr < \infty, \tag{1.5}$$

they proved that (1.1) has an entire large solution if and only if

$$\int_0^\infty r\inf_{|x|=r}\beta(x)dr = \infty. \tag{1.6}$$

They showed that (1.1) has no large solutions if it admits a nontrivial nonnegative bounded solution in Ω.

The literature for the case $p \neq 2$ is less extensive. Existence and nonexistence of large solutions to (1.1), (1.2) have been studied in [1,5,14,18].

For radial continuous function β on $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 3)$ and $h(u) = u^\gamma$, Lu et al. [14] proved the existence of large solutions to (1.1) and (1.2) both for $\gamma > p-1,\ \Omega = \mathbb{R}^N$ or Ω being a bounded domain (superlinear case) and $\gamma \leqslant p-1,\ \Omega = \mathbb{R}^N$ (sublinear case), respectively. When $\gamma \leqslant p-1$ they proved, under the assumption

$$\int_0^\infty (r\phi(r))^{\frac{1}{p-1}} = \infty, \tag{1.7}$$

that (1.1) has an entire large positive radial solution, where

$$\phi(r) = \min_{|x|\leqslant r}\beta(x).$$

They also showed that if (1.1) has an entire large positive solution, then

$$\int_0^\infty (r\rho(r))^{\frac{1}{p-1}}dr = \infty, \tag{1.8}$$

where

$$\rho(r) = \max_{|x|\leqslant r}\beta(x).$$

In [1], Afrouzi et al. considered problems of the kind (1.1), (1.2) for nonnegative nontrivial continuous functions β which are not necessarily radial on $\mathbb{R}^N$. They proved that, under the condition (**H1**), if there exists a positive number ε such that β satisfies

$$\int_0^\infty t^{1+\varepsilon}\sup_{|x|=t}\beta(x)dt < \infty,$$

and $r^{p(N-1)}\sup_{|x|=r}\beta(x)$ is nondecreasing for large r. Then, Eq. (1.1) has a nonnegative nontrivial entire bounded solution. The author also showed that, when β is radially symmetric and $\Omega = \mathbb{R}^N$, if h satisfies the Keller-Osserman condition

$$\int_1^\infty \left(\int_0^s h(t)dt\right)^{\frac{-1}{p}} ds = \infty, \tag{1.9}$$

then (1.1) has a nonnegative nontrivial entire solution. Furthermore, if $r^{p(N-1)}\beta(r)$ is nondecreasing for large r and β staisfies $\int_0^\infty (r\beta(r))^{\frac{1}{p-1}}dr = \infty$, then any nonnegative nontrivial entire solution u to (1.1) is large. Conversely, if (1.1) has a nonnegative entire solution then β satisfies

$$\int_0^\infty r^{1+\varepsilon}\beta(r)dr = \infty$$

for every $\varepsilon > 0$.

Notice that in the former papers, since β is assumed to be radial, the existence of the solution is proved by a fixed-point argument.