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Truncated Lucas sequence and its period

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ABSTRACT

We prove some theorems concerning the truncated Lucas sequences. We also state some conjectures concerning the period (mod m) of truncated Lucas sequences, where the integer m is at least 2. In addition, we present some computer-generated results that tend to support these conjectures.

1. Introduction

We let s_n denote the n^{th} member of the sequence of integers $s_0 = a, s_1 = b, \ldots$, where $s_{n+1} = s_n + s_{n-1}$. A sequence $\{s_i\}$ is said to be *periodic with period* k if it satisfies $s_i = s_{i+k}$ for $i = 1, 2, \ldots$. The symbol $h(m)$ will denote the length of the period of the sequence resulting from reducing each s_n modulo m. An n-step Fibonacci sequence $\{f_k^{(n)}\}_{k=1}^{\infty}$ is defined by letting $f_k^{(n)} = 0$ for $k \leqslant 0$, $f_1^{(n)} = 1$, $f_2^{(n)} = 1$, and other terms according to the linear recurrence equation $f_k^{(n)} = \sum_{i=1}^{n} f_{k-i}^{(n)}$ for $k > 2$. The basic (2-step) Fibonacci sequence will be given by $f_0 = 0$, $f_1 = 1, \ldots$ and the Lucas sequence by $g_0 = 2$, $g_1 = 1, \ldots$. The symbol $k(m)$ will denote the length of the period of the 2- step or 3-step Fibonacci sequences when they are reduced modulo m. We sometimes call it Wall number after a pioneer in this area. We know that $g_n = f_{n-1} + f_{n+1}, n \geqslant 2$. So the Wall number of the Fibonacci sequence is equal to the Wall number of the Lucas sequence.

For example, let f_n be the 2-step Fibonacci sequence and $m = 5$. If we reduce each f_n modulo 5, then we obtain the following periodic sequence.

0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1,... and then repeat; so $k(5) = 20$.

In the recent years, there has been much interest in applications of Fibonacci and Lucas numbers and sequences. Fibonacci and Lucas sequences appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. We can see applications of Fibonacci sequences in group theory in [2,5,8,10] and also see some generalized Fibonacci and Lucas sequences in [9,15,1,3,7,13]. Takahashi gives a fast algorithm which is based on the product of Lucas numbers to compute large Fibonacci numbers [14]. Özkan has proved two theorems concerning the Wall number of the 3-step Fibonacci sequences and given some conjectures concerning $k(m)$ of the 3-step Fibonacci sequence [12]. We will give a few of them in Conjecture 1.3. In [6], Choi et al. introduced the truncated Fibonacci sequences and the truncated Lucas sequences. In [11], the author proved some theorems on truncated Fibonacci sequences and stated some conjectures on their Wall number.

Let $k_r(m)$ denote the Wall number of truncated Fibonacci sequences (defined in Section 2). We will let Z and Z_m represent the group of integers and the group of integers modulo m, respectively.

In this paper, we give some interesting properties of the truncated Lucas sequences and we show that the function $k_r(m)$ associated with this sequence is a multiplicative function. Furthermore, we examine the relationship between m and $k_r(m)$ of the truncated Lucas sequences. We also give results similar to the following theorems. We will give them in Section 3.

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For the 2-step Fibonacci sequences, we gather some of the well-known properties of $k(m)$ into the following two theorems.

Theorem 1.1 [16,17].

i. *(Wall): If m has the prime factorization $m = \prod p_i^{e_i}$ and if $h_i = k(p_i^{e_i})$ then $k(m)$ is the least common multiple of the h_i.*
ii. *(Wall): If $k(p) \neq k(p^2)$, then $k(p^e) = p^{e-1}k(p)$. Also, if n is the largest integer with $k(p) = k(p^n)$, then $k(p^e) = p^{e-1}k(p)$ for $e > n$.*
iii. *(Wilcox): For $m > 2$, $k(m) = 2n$, where $n = \min(\{n : n \text{ even and } m|f_n\} \cup \{n : n \text{ odd and } m|g_n\})$.*

Remark. In all known cases, $k(p^2) = pk(p)$ for p a prime, and it appears to be an open question whether the case $k(p^2) = k(p)$ ever occurs.

Theorem 1.2 [16]. *For the 2-step Fibonacci sequences, the following hold:*

i. *If p is a prime and $p = 10x \pm 1$, then $k(p)|(p-1)$.*
ii. *If p is a prime and $p = 10x \pm 3$, then $k(p)|(2p+2)$.*
iii. *If p is a prime and $p = 10x \pm 3$, then $h(p^e) = k(p^e)$.*

Corollary [16]. *If $p = 10x \pm 3$, then $k(p) \equiv 0(mod4)$.*

Let f_n be the 3-step Fibonacci sequence. Then, $f_n(modm)$ forms *a simply periodic sequence*, that is, the sequence is periodic and repeats by returning to its starting values.

The following conjectures on $k(m)$ appears in [12] ((i)–(iii) concern 3-step Fibonacci sequences). (Note that (iv) does not concern 3-step Fibonacci sequences.)

Conjecture 1.3 [12].

i. *If p is a prime number and $k(p) < p$, then $k(p)|(p-1)$.*
ii. *If p is a prime number and $k(p) \equiv 0(mod2)$, then $(p \pm 1)|k(p)$.*
iii. *If p is a prime number and $k(p) \equiv 1(mod2)$, then $(p \pm 1)|(k(p)-1)$ or $p|(k(p)-1)$.*
iv. *If p is an odd prime number and n is an even number greater than 2, then the Wall number $k(p)$ of the n-step Fibonacci sequence is even.*

2. Truncated Lucas sequence

Throughout this paper, let $\mathbb{R}^m$ denote the infinite dimensional real vector space consisting of all real sequences $(x_0, x_1, x_2, \ldots)^T$, and let P denote The Pascal matrix $[\binom{i}{j}], (i,j = 0,1,2,\ldots)$.

Associated with the Pascal matrix P, let $P^{\rightarrow}P^{\downarrow}$ denote the matrices defined by

$$P^{\rightarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0. & . & . & \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & & . & \\ . & . & . & . & . & . & . & . & & & . \end{bmatrix},$$

where the row i, $(i = 0,1,2,\ldots)$, is that of the Pascal matrix P preceded by 0_i^T, where 0_i is the i-vector of zeros, and

$$P^{\downarrow} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 1 & 1 & 0 & 0 & . & . & . \\ 1 & 2 & 0 & 0 & . & . & . \\ 1 & 3 & 1 & 0 & . & . & . \\ 1 & 4 & 3 & 0 & . & . & . \\ 1 & 5 & 6 & 1 & . & . & . \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix},$$