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An algorithm for autonomously plotting solution sets in the presence of turning points

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ABSTRACT

Plotting solution sets for particular equations may be complicated by the existence of turning points. Here we describe an algorithm which not only overcomes such problematic points, but does so in the most general of settings. Applications of the algorithm are highlighted through two examples: the first provides verification, while the second demonstrates a non-trivial application. The latter is followed by a thorough run-time analysis. While both examples deal with bivariate equations, it is discussed how the algorithm may be generalized for space curves in $\mathbb{R}^3$.

1. Introduction

In this paper we consider curves determined by equations of the form

$$f(x,y) = 0, \tag{1}$$

where $f : I \to \mathbb{R}, I \subset \mathbb{R} \times \mathbb{R}$ is a product of open intervals. Equations such as (1) are often referred to as implicit equations since in general there does not exist an explicit, unique function g such that $y = g(x)$. A canonical example is that of the unit circle, with $f(x,y) = x^2 + y^2 - 1 = 0$, which cannot be rearranged to isolate y as a function of x. The following discussion also applies to equations of the form $f(x;\alpha) = 0$ where α is a bifurcation parameter.

The purpose of this paper is to address the problem of plotting solution curves to (1) with turning points (a precise definition will be presented shortly; see [1–6] for a brief survey). While a method for dealing with this problem has already been developed by Keller [7,8] (via a pseudo-arc-length parametrization), it relies on the turning points to be anything but cusps, and thus cannot be used on curves without some understanding of their profile. Conversely, the algorithm that we will present not only requires no prior knowledge of the solution curve, but also approaches the problem in what we consider to be a more intuitive manner.

It should be said, however, that although we acknowledge that there are other methods to deal with this problem, we will make no attempt to compare them. Essentially, this paper is to be self-contained, and therefore our only concern is the explanation of our proposed algorithm, and its own benefits, not its relative benefits.

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2. Background

Before a discussion of the algorithm may begin, we must first define the term *turning point*.

Definition 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, and $S = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$. A point $(x^*, y^*) \in S$ is a *turning point* of $f(x, y) = 0$ if $(x^*, y^*) \in S$, and there exists $\delta > 0$ such that at least one of the following statements is true:

Type 1: $\{(x, y) \in \mathbb{R}^2 : 0 < x - x^* < \delta,\ 0 < |y - y^*| < \delta\} \cap S = \emptyset$.
Type 2: $\{(x, y) \in \mathbb{R}^2 : 0 < |x - x^*| < \delta,\ 0 < y - y^* < \delta\} \cap S = \emptyset$.
Type 3: $\{(x, y) \in \mathbb{R}^2 : 0 < x^* - x < \delta,\ 0 < |y - y^*| < \delta\} \cap S = \emptyset$.
Type 4: $\{(x, y) \in \mathbb{R}^2 : 0 < |x - x^*| < \delta,\ 0 < y^* - y < \delta\} \cap S = \emptyset$.

Definition 1 may be envisioned in the following way: if one were to restrict themselves to a small enough neighborhood about (x^*, y^*), one of the open "half-balls" centered at (x^*, y^*) contained in this neighborhood would have no intersection with S (see Fig. 1).

The remainder of this paper will deal with the discrete set:

$$S_k = \left\{ \{(x_j, y_j)\}_{j=1}^{N} \subset \mathbb{R}^2 : f_k(x_j, y_j) = 0,\ 1 \leqslant j \leqslant N \right\}, \tag{2}$$

where S_k is the approximation of S, and f_k is the discrete approximation f. For a detailed discussion of approximating turning points in discrete spaces see the review by Cliffe et al. [9].

3. Algorithm

Supposing that $(x_j, y_j) \in S_k$ has been identified as a turning point of any type, let R be the closed disk of radius r centered at (x_j, y_j). The algorithm is then performed in the following steps:

Step 1. Uniformly scan the boundary of R, ∂R, for solutions to $f(x, y) = 0$. Denote the set of solutions as $\mathcal{R}$.

Step 2.

(a) If $\mathcal{R} = \emptyset$, we may stop here: no other points in $\mathbb{R}^2$ satisfy $f(x, y) = 0$.

(b) If $\mathcal{R} \neq \emptyset$, select $(x_i, y_i) \in S_k, i < j$, to be a "reference point" and select the minimal point $(s', t') \in \mathcal{R}$ such that $\phi(s', t') \leqslant \phi(s, t)$ for all $(s, t) \in \mathcal{R}$, where

$$\phi(s, t) = \frac{1}{\sqrt{(x_i - s)^2 + (y_i - t)^2}}. \tag{3}$$

Step 3.

(a) Create the vector $v = (s' - x_j, t' - y_j)$,
(b) Determine the largest axial component of $v, d = \max\{|s' - x_j|, |t' - y_j|\}$.
(c) Use d to determine the new direction of iteration.

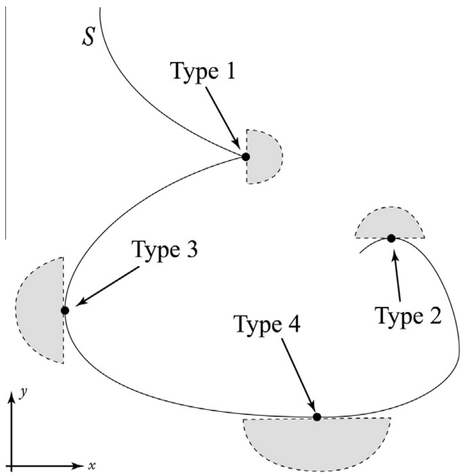

Fig. 1. Illustration of the different types of turning points.