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An improved mapped weighted essentially non-oscillatory scheme

Hui Feng [a], Cong Huang [a], Rong Wang [b,*]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei, PR China
[b] Department of General Education, South University of Science and Technology of China, Shenzhen, Guangdong, PR China

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ABSTRACT

In this paper we develop an improved version of the mapped weighted essentially non-oscillatory (WENO) method in Henrick et al. (2005) [10] for hyperbolic partial differential equations. By rewriting and making a change to the original mapping function, a new type of mapping functions is obtained. There are two parameters, namely A and k, in the new mapping functions (see Eq. (13)). By choosing $k = 2$ and $A = 1$, it leads to the mapping function in Henrick et al., i.e.; the mapped WENO method by Henrick et al. actually belongs to the family of our improved mapped WENO schemes. Furthermore, we show that, when the new mapping function is applied to any $(2r - 1)$th order WENO scheme for proper choice of k, it can achieve the optimal order of accuracy near critical points. Note that, if only one mapping is used, the mapped WENO method by Henrick et al., whose order is higher than five, can not achieve the optimal order of accuracy in some cases. Through extensive numerical tests, we draw the conclusion that, the mapping function proposed by Henrick et al. is not the best choice for the parameter A. A new mapping function is then selected and provides an improved mapped WENO method with less dissipation and higher resolution.

1. Introduction

The essential non-oscillatory (ENO) method was studied by Harten et al. in [6,8,9,7]. It owes its success to the ENO property which is considered to be very important in the numerical simulation for hyperbolic conservation laws because high order numerical methods often produce spurious oscillations, especially near shocks or other discontinuities. The finite-volume ENO spatial discretization was studied by Harten et al. in [7], later the finite-difference ENO method was developed by Shu et al. in [16,17].

Later weighted ENO (WENO) methods have been introduced in [12,11] to address potential numerical instabilities due to the process of choosing ENO stencils [13]. Unlike ENO methods, a convex combination of all the ENO candidate stencils are used in WENO methods such that the numerical flux is approximated with the higher order of accuracy in smooth regions, while WENO methods still retain the ENO property in regions near discontinuities. In this paper we will use WENO-JS to refer the WENO scheme in [11,1]. See [14,15] for an overview and other references.

Recently Henrick et al. [10] noted that the fifth-order WENO method proposed by Jiang and Shu [11] is only third-order accurate near critical points of the smooth regions in general. By using a simple mapping function to the original weights in

* Corresponding author.
E-mail addresses: hui_feng163@tom.com (H. Feng), whiterose003@163.com (C. Huang), wang.r3@sustc.edu.cn (R. Wang).

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[11], Henrick et al. developed a mapped WENO method to achieve the optimal order of accuracy; i.e., the fifth order near critical points. However, when this mapping function is applied to the WENO scheme of the order $\geqslant 7$, it can not achieve the optimal order of accuracy near critical points. In this paper we use WENO-HAP to refer the family of mapped WENO schemes in [11].

Lately Feng et al. [4] studied the mapped WENO method in [10] and found that, when it is used for solving the problems with discontinuities, the mapping function in [10] may amplify the effect from the non-smooth stencils and thus cause a potential loss of accuracy near discontinuities. This effect may be difficult to be observed for the fifth-order WENO method unless a long time simulation is desired. In order to overcome this problem, a new piecewise polynomial mapping function was proposed in [4] by adding two requirements, that is, $g'(0) = g'(1) = 0$ to the original criteria in [10]. Clearly the requirement $g'(0) = 0$ tends to decrease the effect from the non-smooth stencils and $g'(1) = 0$ tends to increase the effect from the smoothest stencil. In this paper we use WENO-PM to refer the family of mapped WENO schemes in [4].

Through a further study of the mapped WENO method, we find that, the mapping function in [10] can be written in a simple and more meaningful form. Make a change to the new form, then a new family of mapping functions, which includes the one in WENO-HAP, is obtained. This kind of new mapping functions still satisfies the original criteria in [10]. Furthermore, when the new mapping function is applied to any $(2r-1)$th order WENO scheme for proper choice of k (see Theorem 2 for more details), it can achieve the optimal order of accuracy near critical points. Note that, if only one mapping is used, the mapped WENO method by Henrick et al., which order is higher than five, can not achieve the optimal order of accuracy in some cases. In order to overcome this problem, two or more mappings need to be used. However, in [4] Feng et al. have pointed out, the more mappings it uses, the less accurate numerical solution it provides due to potential numerical instability. Extensive numerical tests show that, the mapping function in WENO-HAP is not the best mapped WENO method in this new family. Then a new mapping function is selected. It provides an improved mapped WENO method with less dissipation and higher resolution.

The remainder of this paper proceeds as follows. In Section 2, we give a brief description of WENO-JS [11] and WENO-HAP [10]. In Section 3, we give the improved version of the mapped WENO-HAP method. In Section 4, we compare the performance of different WENO methods.

2. The fifth-order WENO method

2.1. WENO spatial discretization

Consider the one-dimensional scalar hyperbolic conservation law

$$u_t = -f_x(u), \quad a < x < b, \; t > 0. \tag{1}$$

Assume a uniform spatial mesh; i.e., $x_j = a + j\Delta x, \; j = 0, 1, \ldots, N$, where $\Delta x = \frac{b-a}{N}$. The spatial derivative $-f_x(u)$ at the point $x = x_j$ is approximated using a conservative finite difference scheme

$$L(u) = -\frac{1}{\Delta x}\left(\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}}\right), \tag{2}$$

where $\hat{F}_{j\pm\frac{1}{2}}$ is the numerical flux at $x_{j\pm\frac{1}{2}}$. The numerical flux must be consistent with f; i.e., $\hat{F}(u, \ldots, u) = f(u)$. The numerical flux determines the particular numerical method and its properties. The fifth-order WENO scheme [11] has a numerical flux which is a convex combination of three third-order fluxes associated with three substencils; i.e.,

$$\hat{F}_{j+\frac{1}{2}} = \omega_0 \hat{F}^0_{j+\frac{1}{2}} + \omega_1 \hat{F}^1_{j+\frac{1}{2}} + \omega_2 \hat{F}^2_{j+\frac{1}{2}}, \tag{3}$$

where

$$\begin{aligned} \hat{F}^0_{j+\frac{1}{2}} &= \tfrac{1}{6}\left(2f_{j-2} - 7f_{j-1} + 11f_j\right), \\ \hat{F}^1_{j+\frac{1}{2}} &= \tfrac{1}{6}\left(-f_{j-1} + 5f_j + 2f_{j+1}\right), \\ \hat{F}^2_{j+\frac{1}{2}} &= \tfrac{1}{6}\left(2f_j + 5f_{j+1} - f_{j+2}\right). \end{aligned} \tag{4}$$

In smooth regions, the WENO approximation is designed such that it approximates the solution with the high order of accuracy. On the other hand, near discontinuities, it keeps the essentially non-oscillatory property of the ENO scheme.

The nonlinear weights are computed as

$$\omega_i = \frac{\alpha_i}{\sum_{j=0}^{2}\alpha_j}, \quad \alpha_i = \frac{d_i}{(\epsilon + IS_i)^2}, \quad i = 0, 1, 2, \tag{5}$$

where ϵ is a small positive number that is introduced to prevent the denominator becoming zero, and the optimal weights, $d_i, i = 0, 1, 2$, are given by

$$d_0 = 1/10, \quad d_1 = 6/10, \quad \text{and} \quad d_2 = 3/10. \tag{6}$$