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A new high order space derivative discretization for 3D quasi-linear hyperbolic partial differential equations

R.K. Mohanty [a,*], Suruchi Singh [b,1], Swarn Singh [c]

[a] Department of Applied Mathematics, South Asian University, Akbar Bhawan, Chanakyapuri, New Delhi 110021, India
[b] Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 11007, India
[c] Department of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi 110021, India

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ABSTRACT

In this paper, we propose a new high accuracy numerical method of $O(k^2 + k^2h^2 + h^4)$ for the solution of three dimensional quasi-linear hyperbolic partial differential equations, where $k > 0$ and $h > 0$ are mesh sizes in time and space directions respectively. We mainly discretize the space derivative terms using fourth order approximation and time derivative term using second order approximation. We describe the derivation procedure in details and also discuss how our formulation is able to handle the wave equation in polar coordinates. The proposed method when applied to a linear hyperbolic equation is also shown to be unconditionally stable. The proposed method behaves like a fourth order method for a fixed value of (k/h^2). Some examples and their numerical results are provided to justify the usefulness of the proposed method.

1. Introduction

We consider the following three dimensional quasi-linear hyperbolic partial differential equation

$$u_{tt} = A(x,y,z,t,u)u_{xx} + B(x,y,z,t,u)u_{yy} + C(x,y,z,t,u)u_{zz} + g(x,y,z,t,u,u_x,u_y,u_z,u_t), \quad 0 < x,y,z < 1, \; t > 0, \tag{1}$$

subject to the initial conditions

$$u(x,y,z,0) = \phi(x,y,z), \quad u_t(x,y,z,0) = \psi(x,y,z), \quad 0 \leqslant x,y,z \leqslant 1 \tag{2}$$

and the boundary conditions

$$u(0,y,z,t) = a_0(y,z,t), \quad u(1,y,z,t) = a_1(y,z,t), \quad 0 \leqslant y, \; z \leqslant 1, \; t \geqslant 0, \tag{3a}$$

$$u(x,0,z,t) = b_0(x,z,t), \quad u(x,1,z,t) = b_1(x,z,t), \quad 0 \leqslant x, \; z \leqslant 1, \; t \geqslant 0, \tag{3b}$$

$$u(x,y,0,t) = c_0(x,y,t), \quad u(x,y,1,t) = c_1(x,y,t), \quad 0 \leqslant x, \; y \leqslant 1, \; t \geqslant 0, \tag{3c}$$

where the Eq. (1) is assumed to satisfy the conditions $A(x,y,z,t,u) > 0$, $B(x,y,z,t,u) > 0$ and $C(x,y,z,t,u) > 0$ in the solution region $\Omega \equiv \{(x,y,z,t) : 0 < x,y,z < 1, \; t > 0\}$. Further we assume that $u(x,y,z,t) \in C^6$, $A(x,y,z,t,u)$, $B(x,y,z,t,u)$, $C(x,y,z,t,u) \in C^4$ and $\phi(x,y,z)$ and $\psi(x,y,z)$ are sufficiently differentiable functions of as higher order as possible.

* Corresponding author.
E-mail addresses: mohantyranjankumar@gmail.com, rmohanty@maths.du.ac.in (R.K. Mohanty).
[1] Present address: Department of Mathematics, Aditi Mahavidyalaya, University of Delhi, New Delhi 110039, India.

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The numerical solution of three dimensional second order quasi-linear hyperbolic equation in Cartesian, cylindrical and spherical coordinates are of great importance in many areas of physics and applied mathematics. Ciment and Leventhal [1,2] have discussed operator compact implicit method to solve wave equation. Using finite element technique, Yoseph et al. [3] have proposed a stable high order method for the solution of multi-dimensional non-linear hyperbolic systems. In 1995, Mohanty et al. [4] have developed a high order numerical method for the solution of three dimensional second order non-linear hyperbolic equation. Later, Mohanty et al. [5] have extended their technique to solve second order quasi-linear hyperbolic equations. In both the cases they have used nine evaluations of the function g and (7 + 19 + 7)-grid points. It has been shown that the linear schemes discussed in [4,5] are conditionally stable. In the recent past, many researchers (see [6–13]) have developed unconditionally stable implicit finite difference methods for the solution of two dimensional linear hyperbolic equations with significant first derivative terms. Most recently, Mohanty and Singh [14,15] have derived high accuracy numerical methods based on Numerov and arithmetic average type discretization for the solution of one and two dimensional quasi-linear hyperbolic equations, in which they have shown that the linear schemes are unconditionally stable. Ding and Zhang [16] have also discussed an unconditionally stable fourth order compact ADI scheme for the two dimensional telegraphic equation.

In this paper, we derive a new compact three-level implicit numerical method of accuracy two in time and four in space for the solution of three dimensional quasi-linear hyperbolic equation (1). We mainly discretize the space derivatives u_x and u_{xx} using fourth order approximation and time derivatives u_t and u_{tt} using second order approximation. In this method, we require only seven evaluations of the function g as compared to nine evaluations of the same function discussed in [4,5]. In the next section, we give formulation of the method. In Section 3, we discuss the application of the proposed method to three dimensional wave equation in polar coordinates, and discuss the stability analysis. In this section, we modify our method in such a way that the solution retains its order and accuracy everywhere in the vicinity of the singularity. In Section 4, we discuss unconditionally stable method for three dimensional telegraphic equation. In Section 5, we examine our new method over a set of linear, non-linear and quasi-linear second order hyperbolic equations whose exact solutions are known and compare the results with the results of other known methods. Concluding remarks are given in Section 6.

2. Numerical algorithm

In this section, we aim to discuss a numerical method for the solution of non-linear wave equation

$$u_{tt} = A(x,y,z,t)u_{xx} + B(x,y,z,t)u_{yy} + C(x,y,z,t)u_{zz} + g(x,y,z,t,u,u_x,u_y,u_z,u_t), \quad 0 < x,y,z < 1,\ t > 0. \tag{4}$$

Let $h > 0$ and $k > 0$ be the mesh spacing in the space and time directions respectively. We replace the solution region $\Omega = \{(x,y,z,t)|0 < x,y,z < 1,\ t > 0\}$ by a set of grid points (x_l, y_m, z_n, t_j) that are denoted by (l,m,n,j), where $x_l = lh$, $l = 0(1)N+1$, $y_m = mh$, $m = 0(1)N+1$, $z_n = nh$, $n = 0(1)N+1$ and $t_j = jk$, $0 < j < J$, N and J being positive integers and $(N+1)h = 1$. Let $\lambda = \frac{k}{h} > 0$ be the mesh ratio parameter. The exact value and the approximate value of u at the grid point (l,m,n,j) are denoted by $U^j_{l,m,n}$ and $u^j_{l,m,n}$, respectively.

At the grid point (l,m,n,j), we denote:

$$W_{pqrs} = \left(\frac{\partial^{p+q+r+s}W}{\partial x^p \partial y^q \partial z^r \partial t^s}\right)^j_{l,m,n}; \quad W = U,A,B,C; \quad p,q,r,s = 0,1,2,\ldots, \tag{5.1}$$

$$\alpha^j_{l,m,n} = \left(\frac{\partial g}{\partial U_x}\right)^j_{l,m,n}, \quad \beta^j_{l,m,n} = \left(\frac{\partial g}{\partial U_y}\right)^j_{l,m,n}, \quad \gamma^j_{l,m,n} = \left(\frac{\partial g}{\partial U_z}\right)^j_{l,m,n}. \tag{5.2}$$

Further, at the grid point (l,m,n,j), the exact solution $U^j_{l,m,n}$ satisfies

$$U_{0002} - A_{0000}U_{2000} - B_{0000}U_{0200} - C_{0000}U_{0020} = g\left(x_l, y_m, z_n, t_j, U^j_{l,m,n}, U^j_{xl,m,n}, U^j_{yl,m,n}, U^j_{zl,m,n}, U^j_{tl,m,n}\right) \equiv G^j_{l,m,n}, \ \text{(say)}. \tag{6}$$

We consider the following approximations:

$$\overline{U}^j_{tl,m,n} = \left(U^{j+1}_{l,m,n} - U^{j-1}_{l,m,n}\right)/(2k), \tag{7.1}$$

$$\overline{U}^j_{tl\pm1,m,n} = \left(U^{j+1}_{l\pm1,m,n} - U^{j-1}_{l\pm1,m,n}\right)/(2k), \tag{7.2}$$

$$\overline{U}^j_{tl,m\pm1,n} = \left(U^{j+1}_{l,m\pm1,n} - U^{j-1}_{l,m\pm1,n}\right)/(2k), \tag{7.3}$$

$$\overline{U}^j_{tl,m,n\pm1} = \left(U^{j+1}_{l,m,n\pm1} - U^{j-1}_{l,m,n\pm1}\right)/(2k), \tag{7.4}$$