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Convex approximations of analytic functions

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ABSTRACT

We introduce a method for constructing the best approximation of an analytic function in a subclass $\mathcal{K}^* \subset \mathcal{K}$ of convex functions, in the sense of the L^2 norm. The construction is based on solving a certain semi-infinite quadratic programming problem, which may be of independent interest.

1. Introduction

Both convexity and injectivity of a function play an important role in many areas of mathematics. In particular, the injectivity of an analytic function (univalency) is an important area of research in the Geometric function theory, and there are many sufficient conditions for univalency in the literature (see for example the monographs [1] or [5]). When an analytic function is not univalent, then in practical problems it is often of interest to find a "best approximation" of it by a univalent function.

In [4] we studied the problem of finding the best approximation of an analytic function in a subclass of the class of starlike univalent functions. Continuing these ideas, in the present paper we consider the problem of finding the best approximation in a subclass $\mathcal{K}^* \subset \mathcal{K}$ of convex univalent functions defined in the unit disk, in the sense of $L^2(U)$ norm. The problem can be reduced to a certain semi-infinite quadratic programming problem, which we solve explicitly in Theorem 4, thus leading to a method for finding the best convex approximation: our main result in Theorem 5 provides a constructive algorithm for finding explicitly the measure $\mathrm{dist}(f, \mathcal{K}^*)$ of the (non) convexity of an analytic function (see also Theorem 3) and and of its best convex approximation.

The paper is structured as follows. In Section 2, we introduce the measures $\mathrm{dist}(f, \mathcal{K})$ and $\mathrm{dist}(f, \mathcal{K}^*)$, showing how far is an analytic function f from the class $\mathcal{K}$ of convex univalent functions, respectively from its subclass $\mathcal{K}^*$ defined by (2). In Theorem 3 we show that $\mathrm{dist}(f, \mathcal{K}) = 0$ iff $f \in \mathcal{K}$, so $\mathrm{dist}(f, \mathcal{K})$ is indeed a measure of the (non) convexity of the function f (a similar result holds for $\mathrm{dist}(f, \mathcal{K}^*)$).

Next, Lemma 2 and the definitions of the class $\mathcal{K}^*$ and $\mathrm{dist}(f, \mathcal{K}^*)$ lead us to consider the semi-infinite quadratic problem (8), (9) (infinite number of variables, finite number of constraints). In Theorem 4 we solve this problem, determining explicitly the minimum value of the objective function and the extremum point. The idea of the proof is to show that the Karush–Kuhn–Tucker conditions can be applied in this infinite-dimensional setting (see Remark 3), and to use a certain rearrangement of the coefficients of the problem which allows us to solve it explicitly.

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As an application of Theorem 4, in Theorem 5 we obtain the best convex approximation of a normed analytic function defined in the unit disk. The result gives explicitly the value of $\operatorname{dist}(f, \mathcal{K}^*)$ and of the extremal convex function, and is thus suitable for both numerical implementation and applications.

The paper concludes with two examples, which show that when $\operatorname{dist}(f, \mathcal{K}^*)$ is not too large, the method of Theorem 5 produces a good convex approximation of a given analytic function $f \in \mathcal{A}$ (see Fig. 1).

2. Main results

We will denote by $\mathcal{A}$ the class of analytic functions $f : U \to \mathbb{C}$ satisfying the normalization condition $f(0) = f'(0) - 1 = 0$, and by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of convex univalent functions in the unit disk U (functions $f \in \mathcal{A}$ that map U univalently onto a convex domain).

It is known (see for example [2]) that if $f \in \mathcal{A}$ has the series development

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{C}, \tag{1}$$

and the coefficients a_n satisfy the inequality

$$\sum_{n=2}^{\infty} n^2 |a_n| \leqslant 1 \tag{2}$$

then $f \in \mathcal{K}$. We denote by $\mathcal{K}^*$ the subclass of $\mathcal{K}$ defined by (2).

As a measure of non-univalency of a function $f \in \mathcal{A}$, in [4] we introduced the following:

$$\operatorname{dist}(f, \mathcal{U}) = \inf_{g \in \mathcal{U}} \left(\int_U |f(x+iy) - g(x+iy)|^2 dxdy \right)^{1/2}, \tag{3}$$

and we showed that $\operatorname{dist}(f, \mathcal{U}) = 0$ iff $f \in \mathcal{U}$, so $\operatorname{dist}(f, \mathcal{U})$ is a measure showing how "far" is the function f from being univalent. Similarly, as a measure of the (non) convexity of a function, we introduce the following.

Definition 1. For $f \in \mathcal{A}$ we define

$$\operatorname{dist}(f, \mathcal{K}) = \inf_{g \in \mathcal{K}} \left(\int_U |f(x+iy) - g(x+iy)|^2 dxdy \right)^{1/2}, \tag{4}$$

with a similar definition for $\operatorname{dist}(f, \mathcal{K}^*)$.

Using Fubini's theorem and the orthogonality of the trigonometric functions, the integral of $|f|^2$ over U can be expressed in term of the coefficients of the Taylor series of f, as follows.

Lemma 2. ([4]) *If $f : U \to \mathbb{C}$ is analytic in U and has series development*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U, \tag{5}$$

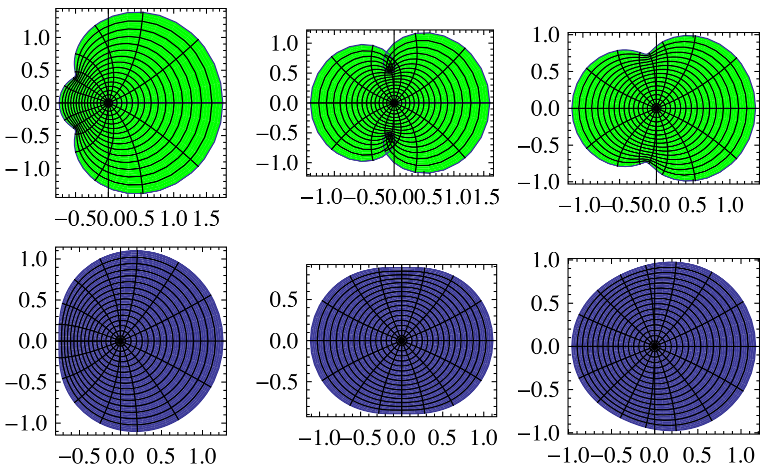

Fig. 1. The image of the unit disk under $f_{a,b}$ (top row) and $g_{a,b}$ (bottom row), for $(a, b) = (0.5, 0.25)$ (left), $(a, b) = (0.1, 0.5)$ (center), and $(a, b) = (0.25, 0.5)$ (right).