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Lyapunov-type inequality for higher order difference equations

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ABSTRACT

In this paper, a new Lyapunov-type inequality for a class of higher order difference equations with anti-periodic boundary conditions is established. Applying our inequality, the lower bound of the eigenvalue for a related eigenvalue problem is given.

1. Introduction

The Lyapunov inequality and many of its generalizations have proven to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations [1–6]. Although there is an extensive literature on the Lyapunov-type inequalities for various classes of differential equations, there is not much done for the difference equations, especially, for the higher order difference equations. The discrete and time scale analogues of Lyapunov-type inequalities for certain type systems are also given in [7]. In 1983, Cheng [8] investigated the following second-order difference equation

$$\triangle^2 x(n) + q(n)x(n+1) = 0. \tag{1}$$

Under the following assumptions

$$x(a) = x(b) = 0, \quad x(n) \not\equiv 0, \quad n \in Z[a,b], \tag{2}$$

where $a, b \in N, Z[a,b] = \{a, a+1, \ldots, b-1, b\}$, they [8] obtained the following Lyapunov inequality

$$\mathcal{F}(b-a)\sum_{n=a}^{b-2} q(n) \geqslant 4, \tag{3}$$

where

$$\mathcal{F}(m) = \begin{cases} \frac{m^2-1}{m}, & \text{if m} - 1 \text{ is even}, \\ m, & \text{if m} - 1 \text{ is odd}. \end{cases} \tag{4}$$

In 2012, Zhang and Tang [9] considered the following even order difference equation

$$\triangle^{2k} x(n) + (-1)^k q(n)x(n+1) = 0, \tag{5}$$

where $k \in N, n \in Z$ and $q(n)$ is a real-valued function defined on Z. Under the following boundary conditions

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$$\triangle^{2i}x(a) = \triangle^{2i}x(b) = 0, \quad x(n) \not\equiv 0, \quad n \in Z[a,b], \tag{6}$$

where $a, b \in N, Z[a,b] = \{a, a+1, \ldots, b-1, b\}$, they [9] obtained the following main result:

Theorem A. *Assume that $k \in N$ and $q(n)$ is a real-valued function on Z. If (5) has a solution $x(n)$ satisfying the boundary value conditions (6), then*

$$\sum_{n=a}^{b-1}[|q(n)|(n-a+1)(b-n-1)] \geqslant \frac{2^{3(k-1)}}{(b-a)^{2k-3}}. \tag{7}$$

Recently, anti-periodic problems have received considerable attention as anti-periodic boundary conditions appear in numerous situations such as anti-periodic trigonometric polynomials in the study of interpolation problems and anti-periodic wavelets. Wang and Shi [10] investigated the existence of eigenvalues of second-order difference equations with periodic and anti-periodic boundary conditions. Wang [11] established Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions. To the best of our knowledge, few authors have discussed the Lyapunov-type inequalities for higher order difference equations under anti-periodic boundary conditions. In this paper, we will consider the following higher order difference equation

$$|\triangle^m x(n)|^{p-2}\triangle^m x(n) + r(n)|x(n)|^{p-2}x(n) = 0, \tag{8}$$

where $m \in N, n \in Z, r(n)$ is a real-valued function defined on $Z, p > 1$ and q is a conjugate exponent, i.e. $1/p + 1/q = 1$.

We will establish a new discrete Lyapunov-type inequality for (8) under the following anti-periodic boundary conditions

$$\triangle^i x(a) + \triangle^i x(b) = 0, \quad i = 0, 1, \ldots, m-1; \quad x(n) \not\equiv 0, \ n \in Z[a,b]. \tag{9}$$

Furthermore, applying our Lyapunov-type inequality to the related eigenvalue problem, we will give the lower bound of the eigenvalue.

2. Main results

Theorem 1. *If (8) has a nonzero solution $x(n)$ satisfying the anti-periodic boundary conditions (9), then*

$$\sum_{n=a}^{b-1}|r(n)|^q \geqslant \frac{2^{mp}}{(b-a)^{(mp-1)}}.$$

Proof. Since the nonzero solution $x(n)$ of (8) satisfies the anti-periodic boundary conditions (9), then $x(a) + x(b) = 0$. For $n \in Z[a,b]$, we have

$$x(n) = x(n) - \frac{1}{2}[x(a) + x(b)] = \frac{1}{2}\sum_{k=a}^{n-1}[x(k+1) - x(k)] - \frac{1}{2}\sum_{k=n}^{b-1}[x(k+1) - x(k)] = \frac{1}{2}\sum_{k=a}^{n-1}\triangle x(k) - \frac{1}{2}\sum_{k=n}^{b-1}\triangle x(k). \tag{10}$$

Using discrete Hölder inequality gives

$$|x(n)| \leqslant \frac{1}{2}\sum_{k=a}^{b-1}|\triangle x(k)| \leqslant \frac{1}{2}(b-a)^{1/q}\left(\sum_{k=a}^{b-1}|\triangle x(k)|^p\right)^{1/p}. \tag{11}$$

Similarly,

$$|\triangle^i x(n)| \leqslant \frac{1}{2}\sum_{k=a}^{b-1}|\triangle^{i+1}x(k)| \leqslant \frac{1}{2}(b-a)^{1/q}\left(\sum_{k=a}^{b-1}|\triangle^{i+1}x(k)|^p\right)^{1/p}. \tag{12}$$

Then

$$|\triangle^i x(n)|^p \leqslant \left(\frac{1}{2}\right)^p (b-a)^{p/q}\left(\sum_{k=a}^{b-1}|\triangle^{i+1}x(k)|^p\right). \tag{13}$$

Summing (13) from a to $b-1$, we have

$$\sum_{n=a}^{b-1}|\triangle^i x(n)|^p \leqslant (b-a)\left(\frac{1}{2}\right)^p (b-a)^{p/q}\left(\sum_{k=a}^{b-1}|\triangle^{i+1}x(k)|^p\right), \tag{14}$$