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Numerical approximation of a nonlinear fourth-order integro-differential equation by spectral method

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ABSTRACT

A spectral method is developed to approximate a nonlinear fourth-order integro-differential equation. An iterative algorithm is proposed to solve the discrete algebraic system. Convergence analysis of the iteration is carried out. An error estimate is also derived for the proposed method. Several numerical examples are presented to confirm the efficiency and accuracy of the overall algorithm.

1. Introduction

The following nonlinear fourth-order integro-differential equation arises from the study of transverse vibrations of a hinged beam:

$$\begin{cases} y^{(4)} - \varepsilon y'' - \frac{2}{\pi}\left(\int_0^{\pi} (y')^2 dx\right) y'' = p(x), \quad 0 < x < \pi, \\ y(0) = y(\pi) = y''(0) = y''(\pi) = 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a constant independent of any function, and $p(x)(\leqslant 0 \text{ or } \geqslant 0)$ is a continuous function on $[0, \pi]$. We refer to [4,12] for a derivation of the above equation. Numerical methods devoted to solve Eq. (1.1) have been widely developed and studied by many researchers. For example, finite element approximation and error estimates of this equation were given in [6]. An iterative finite element method motivated by [7,8] was considered for this equation in [11]. Numerical results showed that the iterative method generate fast convergent iterative process for the original problem. A finite difference method together with Newton type iterative algorithm was introduced to approximate the equation in [3].

We notice that in the mentioned research work, high degree of freedom is needed to get high accuracy. Hence, high-order methods are particularly preferable for Eq. (1.1) over low-order methods.

Spectral methods are high-order methods and powerful tools for solving many kinds of differential and integro-differential equations (see [1,2,10] and the references therein). Considering the high accuracy of spectral methods, it is natural for us to introduce the spectral approximation on the fourth-order integro-differential equation (1.1). We will first rewrite the equation into a coupled system of two second-order equations. Then, Legendre–Galerkin spectral method together with the iterative algorithm proposed in [11] will be applied to approximate the coupled system. By constructing appropriate basis functions for the approximation space, the coupled system is reduced to a discrete algebraic system with sparse structured matrices, which can be efficiently solved on each iterative step.

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This paper is organized as follows. In Section 2, we describe the Legendre–Galerkin spectral approximation of the model fourth-order integro-differential equation. An iterative algorithm is also proposed to solve the discrete algebraic system. Rigorous convergence analysis of the iteration is carried out in Section 3. In Section 4, error estimate of the method is established. Several numerical examples are given in Section 5 to confirm the theoretical results. A short conclusion of this paper is given in the last section.

2. Legendre–Galerkin spectral approximation

The main objective of this section is to propose the spectral approximation on the model problem and to describe the iterative technique motivated by [7,8] on the discretized algebraic system. Let $\Lambda = (0, \pi)$. We denote the inner product of $L^2(\Lambda)$ by $(u, v)_\Lambda = \int_\Lambda u(x)v(x)dx$, whose norm is denoted by $\|u\|_\Lambda = (u, u)_\Lambda^{\frac{1}{2}}$. For any nonnegative integer $r, H^r(\Lambda)$ is the usual Sobolev space, whose norm and semi-norm are defined as follows

$$|v|_{r,\Lambda} = \|v^{(r)}\|_\Lambda, \quad \|v\|_{r,\Lambda} = \left(\sum_{k=0}^{r} |v|^2_{k,\Lambda}\right)^{\frac{1}{2}},$$

where $v^{(r)}$ is the rth order derivative of function $v(x)$. Moreover, $H_0^1(\Lambda)$ is the subspace of $H^1(\Lambda)$, defined by

$$H_0^1(\Lambda) = \{v \in H^1(\Lambda) : v(0) = v(\pi) = 0\}.$$

In cases where no confusion would arise, Λ may be dropped from the notations.

Let $\phi(x) = -y''(x)$. Then, Eq. (1.1) can be rewritten as

$$\begin{cases} -\phi'' + \varepsilon\phi + \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right)\phi = p(x), & 0 < x < \pi, \\ \phi(0) = \phi(\pi) = 0, & \\ -y'' - \phi = 0, & 0 < x < \pi, \\ y(0) = y(\pi) = 0. & \end{cases} \tag{2.1}$$

The weak formulation of (2.1) is to find $(\phi, y) \in H_0^1(\Lambda) \times H_0^1(\Lambda)$ such that

$$\begin{cases} (\phi', \varphi') + \varepsilon(\phi, \varphi) + \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right)(\phi, \varphi) = (p, \varphi), & \forall \varphi \in H_0^1(\Lambda), \\ (y', \eta') - (\phi, \eta) = 0, & \forall \eta \in H_0^1(\Lambda). \end{cases} \tag{2.2}$$

Now, we set

$$X_N = \text{span}\{L_0(x), L_1(x), \ldots, L_N(x)\}, \quad X_N^0 = X_N \cap H_0^1(\Lambda),$$

where $L_i(x)$ is the Legendre polynomial with degree i. Then, the Legendre–Galerkin spectral approximation to (2.2) is as follows:

Find $(\phi_N, y_N) \in X_N^0 \times X_N^0$ such that

$$\begin{cases} (\phi_N', \varphi_N') + \varepsilon(\phi_N, \varphi_N) + \frac{2}{\pi}\left(\int_0^\pi (y_N')^2 dx\right)(\phi_N, \varphi_N) = (p, \varphi_N), & \forall \varphi_N \in X_N^0, \\ (y_N', \eta_N') - (\phi_N, \eta_N) = 0, & \forall \eta_N \in X_N^0. \end{cases} \tag{2.3}$$

Throughout this paper, we will assume that $p(x) \leqslant 0$. We will obtain a similar result in the case $p(x) \geqslant 0$.

The actual algebraic system for (2.3) will depend on the basis functions of X_N^0. To construct suitable basis functions for the approximation space, we will use the orthogonality of the Legendre polynomials. First, we transform the physical domain Λ to the reference domain $\hat{\Lambda} = (-1, 1)$ by the coordinate transformation,

$$\hat{x} := \hat{x}(x) = \frac{2}{\pi}x - 1, \quad \forall x \in \Lambda.$$

Then, following [9], we define for $j = 0, 1, \ldots, N-2$,

$$\psi_j(x) = c_j\left(L_j(\hat{x}) - L_{j+2}(\hat{x})\right), \quad x \in \Lambda,$$

where $c_j = \frac{1}{\sqrt{4j+6}}$. It is readily seen that

$$X_N^0 = \text{span}\{\psi_0(x), \psi_1(x), \ldots, \psi_{N-2}(x)\}.$$

We now turn to derive the matrix statement of (2.3). Let $\gamma_k = \frac{2}{2k+1}$, $a_{ij} = (\psi_j', \psi_i')$ and $b_{ij} = (\psi_j, \psi_i)$. Thanks to the orthogonality of the Legendre polynomials, we obtain that

$$a_{ij} = a_{ji} = \begin{cases} \frac{2}{\pi}, & j = i, \\ 0, & j \neq i, \end{cases} \quad b_{ij} = b_{ji} = \frac{\pi}{2} \times \begin{cases} c_i^2(\gamma_i + \gamma_{i+2}), & j = i, \\ -c_{i+2}c_i\gamma_{i+2}, & j = i+2, \\ 0, & \text{otherwise}. \end{cases}$$