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Homoclinic solutions of discrete nonlinear Schrödinger equations with asymptotically or super linear terms ☆

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ABSTRACT

In this paper, we study the following periodic discrete nonlinear Schrödinger equation

$$-\Delta u_n + (\epsilon_n - \omega)u_n = \gamma\chi_n g_n(u_n), \quad n \in \mathbb{Z},$$

where the nonlinearity g_n is asymptotically or super linear. The sufficient conditions on the existence and on the nonexistence of nontrivial solitons was established by using critical point theory.

1. Introduction and main result

Let us consider the following periodic discrete nonlinear Schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + \epsilon_n\psi_n - \gamma\chi_n f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, $\gamma = \pm 1$, $\{\epsilon_n\}$ and $\{\chi_n\}$ are positive real valued N-periodic sequences ($N > 0$), the nonlinearity f_n is supposed to be a gauge invariant complex-valued function of complex variable, i.e., $f_n(\psi_n e^{-i\omega t}) = e^{-i\omega t} f_n(\psi_n)$ for any $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$.

Making use of the standing wave ansatz

$$\psi_n = u_n e^{-i\omega t},$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency, we arrive at the equation

$$-\Delta u_n + V_n u_n = \gamma\chi_n f_n(u_n), \quad n \in \mathbb{Z},$$

where $V_n := \epsilon_n - \omega$. In fact, we shall consider the following more general equation

$$-\Delta u_n + V_n u_n = \gamma\chi_n g_n(u_n), \quad n \in \mathbb{Z}, \tag{1.2}$$

where g_n is a sequence of general functions. As usual, solitons (that is homoclinic solutions) of (1.1) are spatially localized time-periodic solutions and decay to 0 at infinity, that is,

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$$\lim_{|n|\to\infty} u_n = 0. \tag{1.3}$$

In addition, u is called a nontrivial solitons if $u_n \not\equiv 0$. Therefore, the problem on the existence of nontrivial solitons of (1.1) has been reduced to that on the existence of nontrivial solitons of (1.2) with $g_n \equiv f_n$.

Let $G_n(s) := \int_0^s g_n(t)dt,\ s \in \mathbb{R}$. We make the following assumptions:

($\mathbf{G_1}$) $g_n(s)$ is continuous in s, $g_n(\cdot) = g_{n+N}(\cdot)$ and $G_n(s) \geqslant 0$ for all $(n,s) \in \mathbb{Z}\times\mathbb{R}$.
($\mathbf{G_2}$) There exist some $c > 0$ and $p > 2$ such that $|g_n(s)| \leqslant c(1+|s|^{p-1})$ for all $(n,s) \in \mathbb{Z}\times\mathbb{R}$.
($\mathbf{G_3}$) $g_n(s) = o(s)$ as $s \to 0$ for all $n \in \mathbb{Z}$.
($\mathbf{G_4}$) $\frac{g_n(s)}{s} \to a$ as $|s| \to +\infty$ for all $n \in \mathbb{Z}$, where $0 < a \leqslant +\infty$.

Remark 1.1. Assumption (G_2) will not be needed in the *asymptotically linear* case (that is, $0 < a < +\infty$ in (G_4)), since observe that (G_2) always hold when $0 < a < +\infty$ in (G_4).

Let $\underline{\chi} := \min_{n\in\mathbb{Z}}\{\chi_n\},\ \overline{\chi} := \max_{n\in\mathbb{Z}}\{\chi_n\}$, $\underline{V} := \min_{n\in\mathbb{Z}}\{\epsilon_n\} - \omega$ and $\overline{V} := \max_{n\in\mathbb{Z}}\{\epsilon_n\} - \omega$. In what follows, we always assume that

- $0 < \underline{V} \leqslant V_n \leqslant \overline{V} < +\infty.$

Let $H_n(s) := \frac{1}{2}g_n(s)s - G_n(s),\ s \in \mathbb{R}$. Firstly, we handle the *asymptotically linear* problem, that is, $a < +\infty$ in (G_4) holds. We use the following assumption:

($\mathbf{A_1}$) $H_n(s) \geqslant 0$ for all $(n,s) \in \mathbb{Z}\times\mathbb{R}$ and there is $\zeta \in (0, \underline{V}/2\overline{\chi}]$ such that

$$\frac{g_n(s)}{s} \geqslant \underline{V}/2\overline{\chi} - \zeta \Rightarrow H_n(s) \geqslant \zeta.$$

Next, we deal with the *super linear* case, that is, $a = +\infty$ in (G_4) holds. We make the following assumption:
($\mathbf{A_2}$) There exists $D \in [1,\infty)$ such that $H_n(s) \leqslant DH_n(t)$ for all $(s,t) \in \mathbb{R}\times\mathbb{R}$ with $|s| \leqslant |t|$.

Example 1.1 (*Asymptotically linear case*). Let

$$g_n(s) = b_n s\left(1 - \frac{1}{\ln(e+|s|)}\right), \quad s \in \mathbb{R},$$

where $b_n = b_{n+N}$ and $\min_{n\in\mathbb{Z}} b_n > 0$. It is not hard to check that this function satisfies (G_1)–(G_4) with $0 < a < +\infty$ and (A_1).

Example 1.2 (*Super linear case*). Let

$$g_n(s) = a_n|s|^{p-2}s, \quad s \in \mathbb{R},$$

where $a_n = a_{n+N}, 0 < \min_{n\in\mathbb{Z}} a_n \leqslant \max_{n\in\mathbb{Z}} a_n < \infty$ and $p > 2$. It is not hard to check that this function satisfies $(G_1) - (G_4)$ with $a = +\infty$ and (A_2).

In this paper, we focus our main attention on the existence and the nonexistence of nontrivial solitons for the periodic discrete nonlinear Schrödinger equations with asymptotically or super linear terms. In what follows, we always assume that $\gamma = 1$. The other case reduces to the previous one if we replace $\triangle$ by $-\triangle$ and V_n by $-V_n$. Now, our main results read as follows:

Theorem 1.1. *Assume that* (G_1)–(G_4) *and* $0 < \underline{V} \leqslant V_n \leqslant \overline{V} < \infty$ *for all* $n \in \mathbb{Z}$ *hold. If either* $(0 < a < +\infty,\ 4 + \overline{V} < a\underline{\chi}$ *and* $(A_1))$ *or* $(a = +\infty$ *and* $(A_2))$ *hold, then there exists a nontrivial solution of (1.2).*

Let L is a Jacobi operator [25] given by $Lu_n := -\triangle u_n + \epsilon_n u_n$, then we have the following theorem:

Theorem 1.2. *Assume that the conditions of Theorem 1.1 hold. If* $\omega \notin \sigma(L)$, *then the solution* u *of (1.2) decays exponentially at infinity, that is, there exist two positive constants* τ *and* ν *such that*

$$|u_n| \leqslant \tau e^{-\nu|n|}, \quad n \in \mathbb{Z}. \tag{1.4}$$

Remark 1.2. If the temporal frequency $\omega \in \sigma(L)$, arguing as in [18], we can prove that (1.2) has no well-decaying (e.g. exponentially fast) nontrivial solution.

Theorem 1.3. *Assume that* (G_4) *with* $0 < a < +\infty$ *holds, moreover,*