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Gauss–Markov processes in the presence of a reflecting boundary and applications in neuronal models

A. Buonocore [a], L. Caputo [a], A.G. Nobile [b], E. Pirozzi [a,*]

[a] Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Monte S. Angelo 80126, Napoli, Italy
[b] Dipartimento di Studi e Ricerche Aziendali (Management & Information Technology), Università di Salerno, Via Giovanni Paolo II, n. 132, 84084 Fisciano, SA, Italy

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ABSTRACT

Gauss–Markov processes restricted from below by special reflecting boundaries are considered and the transition probability density functions are determined. Furthermore, the first-passage time density through a time-dependent threshold is studied by using analytical, numerical and asymptotic methods. The restricted Gauss–Markov processes are then used to construct inhomogeneous leaky integrate-and-fire stochastic models for single neurons activity in the presence of a reversal hyperpolarization potential and time-varying input signals.

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1. Introduction

Diffusion and Gauss–Markov processes have been increasingly used as approximations to the description of the evolution of systems such as natural populations, queues and neuronal systems (see, for instance, [2,6,7,9,10,24,26]). In particular, in studies of population dynamics, with immigration effects, and in queueing systems, the number of individuals or customers is bound to take non negative values, so that a reflection condition at zero must thus be imposed. Instead, in neuronal modeling the membrane potential evolution can be described by focusing the attention on stochastic processes confined by a reflecting boundary that can be looked at as the neuronal reversal hyperpolarization potential.

In the previous types of instances, first-passage time densities are to be invoked to describe events such as the extinction of populations and the emptying of queues (corresponding to the attainment of close-to-zero sizes) and neuronal firings, that originate when a suitable threshold value is reached by the modeled time-course of the membrane potential. However, usually the knowledge of the free transition probability density function (pdf) is not sufficient to determine the transition pdf in the presence of preassigned reflecting time-dependent boundaries. Such densities are actually known only in few special cases, such as the Wiener process, whose transition pdf in the presence of a reflecting constant boundary is obtained by the method of images (cf., for instance, [5]). In [14] time non-homogeneous diffusion processes, confined by a time dependent reflecting boundary have been investigated, to obtain a system of integral equations concerning the transition pdf in the presence of the reflecting boundary.

In the present paper, closed form solutions for the transition pdf will be derived for Gauss–Markov processes restricted by particular time dependent reflecting boundaries. Furthermore, the first-passage time (FPT) problem through time-dependent thresholds will be analyzed and a non-singular second-kind Volterra integral equation for the FPT pdf will be obtained. A

* Corresponding author.
E-mail addresses: anibuono@unina.it (A. Buonocore), luigia.caputo@unina.it (L. Caputo), nobile@unisa.it (A.G. Nobile), epirozzi@unina.it (E. Pirozzi).

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number of closed form results will be explicitly stated in view of their potential applications to various fields including neuronal modeling.

In particular, in Section 2 a Gauss–Markov process restricted from below by a special reflecting boundary is considered and the transition probability density function, the conditional mean and the second order conditional moment are determined. For such a process, in Section 3 the first-passage time to a time-dependent threshold is studied by using analytical, numerical and asymptotic methods. Furthermore, in Section 4 inhomogeneous leaky integrate-and-fire (LIF) stochastic models for single neurons activity in the presence of a reversal hyperpolarization potential and time-varying input signals are considered.

2. Restricted Gauss–Markov processes

Let $m(t), h_1(t), h_2(t)$ be $C^1(T)$-class functions (with T continuous parameter set), such that $h_2(t) \neq 0$ and $r(t) = h_1(t)/h_2(t)$ is a non-negative and monotonically increasing function. Denoting by $\{W(t), t \geqslant 0\}$ a standard Wiener process, then

$$Y(t) = m(t) + h_2(t)\, W[r(t)] \tag{1}$$

is a non-singular Gauss–Markov process with mean $m(t)$ and covariance $c(s,t) = h_1(s)\, h_2(t)$ for $s \leqslant t$. The free transition pdf $f_Y(x,t|y,\tau)$ of $Y(t)$ is a normal density having mean and variance (cf. [8,22]):

$$\begin{aligned} M(t|y,\tau) &= E[Y(t)|Y(\tau)=y] = m(t) + \frac{h_2(t)}{h_2(\tau)}\,[y - m(\tau)], \\ V(t|\tau) &= \mathrm{Var}[Y(t)|Y(\tau)=y] = h_2(t)\left[h_1(t) - \frac{h_2(t)}{h_2(\tau)} h_1(\tau)\right] \end{aligned} \tag{2}$$

for $t, \tau \in T$ and $\tau < t$.

We now construct a new stochastic process $\{X(t), t \in T\}$, defined in $[v(t), +\infty)$, obtained by considering $\{Y(t), t \in T\}$ in the presence of a reflecting lower boundary $v(t) \in C^1(T)$. For $t > \tau$, the sample paths of $X(t)$ are confined in $[v(t), +\infty)$. The transition pdf $f_X(x,t|y,\tau)$ in the presence of the reflecting boundary $v(t)$ can be obtained as the solution of the Fokker–Planck equation

$$\frac{\partial f_X(x,t|y,\tau)}{\partial t} = -\frac{\partial}{\partial x}\,[A_1(x,t) f_X(x,t|y,\tau)] + \frac{A_2(t)}{2}\,\frac{\partial^2 f_X(x,t|y,\tau)}{\partial x^2} \quad [x > v(t), y > v(\tau)], \tag{3}$$

with the delta initial condition

$$\lim_{t \downarrow \tau} f_X(x,t|y,\tau) = \delta(x-y) \tag{4}$$

and the additional requirement that a reflecting condition is imposed on the boundary $v(t)$, i.e.,

$$\lim_{x \downarrow v(t)} \left\{ A_1(x,t) f_X(x,t|y,\tau) - \frac{A_2(t)}{2}\,\frac{\partial f_X(x,t|y,\tau)}{\partial x} \right\} - v'(t) f_X[v(t),t|y,\tau] = 0. \tag{5}$$

The functions $A_1(x,t)$ and $A_2(t)$ in (3) are the infinitesimal moments of the process $Y(t)$, given by

$$A_1(x,t) = m'(t) + [x - m(t)]\,\frac{h_2'(t)}{h_2(t)}, \quad A_2(t) = h_2^2(t)\, r'(t), \tag{6}$$

the prime denoting the derivative with respect to the argument. Since $v(t)$ is a reflecting boundary, the total probability mass of $X(t)$ is conserved in $[v(t), +\infty)$, i.e.,

$$\int_{v(t)}^{+\infty} f_X(x,t|y,\tau)\, dx = 1 \quad [y \geqslant v(\tau)]. \tag{7}$$

By choosing the boundary $v(t)$ in a suitable way, it is possible to determine $f_X(x,t|y,\tau)$ in closed form as the following proposition shows.

Proposition 2.1. *Let $\{Y(t), t \in T\}$ be a non singular Gauss–Markov process having mean $m(t)$ and covariance $c(s,t) = h_1(s)h_2(t)$ for $s \leqslant t$. Then, the transition pdf of the stochastic process $\{X(t), t \in T\}$, obtained by considering $Y(t)$ in presence of the reflecting boundary*

$$v(t) = m(t) + A\, h_1(t) + B\, h_2(t) \quad (A, B \in \mathbb{R}, t \in T), \tag{8}$$

is:

$$f_X(x,t|y,\tau) = f_Y(x,t|y,\tau) - \frac{\partial}{\partial x}\left[\exp\left\{-\frac{2A}{h_2(t)}\,[x - v(t)]\right\} F_Y[2\, v(t) - x, t|y,\tau]\right] \quad [x \geqslant v(t),\ y \geqslant v(\tau)], \tag{9}$$

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