Applied Mathematics and Computation 232 (2014) 820–835

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Computational algebraic geometry and global analysis of regional manipulators

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ARTICLE INFO

Keywords:
Regional manipulators
Kinematical analysis
Algebraic geometry
Gröbner bases
Ideal decomposition

ABSTRACT

The global analysis of the singularities of regional manipulators is addressed in this paper. The problem is approached from the point of view of computational algebraic geometry. The main novelty is to compute the syzygy module of the differential of the constraint map. Composing this with the differential of the forward kinematic map and studying the associated Fitting ideals allows for a complete stratification of the configuration space according to the corank of singularities. Moreover using this idea we can also compute the boundary of the image of the forward kinematic map. Obviously this gives us also a description of the image itself, i.e. the manipulator workspace. The approach is feasible in practice because generators of syzygy modules can be computed in a similar way as Gröbner bases of ideals.

1. Introduction

Robot manipulators inevitably possess forward kinematic singularities. It is therefore desirable to ensure at least that a manipulator does not possess higher-order forward kinematic singularities. Since this is known to be a generic property [1], such manipulators are called generic. That is, typically their set of forward kinematic singularities is a smooth manifold. Most industrial manipulators are non-generic, however, in the sense that their singular variety is not a smooth manifold. The analysis of kinematic singularities of serial manipulators is commonly attempted by means of local approaches [2,3]. The stratification of the singular set according to the corank of singularities of regional manipulators was introduced by Pai and Leu [1]. Global analysis was only presented for particular regional manipulators [4–6].

The kinematics of a closed loop mechanism is described by a system of constraint equations that define its configuration space (c-space) – the variety generated by the constraints. This c-space is in general not a smooth manifold but possesses singular points, which correspond to points where the mechanism's instantaneous mobility changes. Hence the singularities of the c-space variety are critical configurations that interfere with the operation of the mechanism. The kinematics of a serial manipulator (i.e. a serial kinematic chain) is described by its forward kinematics mapping. Its configuration space is a priori a smooth manifold. Serial manipulators exhibit forward kinematic singularities, which are merely the critical points of the forward kinematic mapping forming a subvariety Σ within the c-space. From an application point of view it is imperative to characterize the singular variety in regard to its manifold structure. In particular if Σ is a smooth manifold, then the manipulator's motion is well-defined when passing through a singularity. Since the singular variety is also defined by the

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http://dx.doi.org/10.1016/j.amc.2014.01.138
0096-3003/

vanishing of certain polynomials, the singularities of closed loop and serial chains can be treated in a uniform way, as shown in this paper.

Traditionally it is common practice to model the kinematics of serial manipulators by means of relative coordinates, i.e. using joint angles as configuration parameters. Aiming at a general approach also covering closed loop mechanisms in this paper the so called absolute coordinate formulation (spatial position and orientation of each link) is used to construct the forward kinematic mapping. This formulation applies equally to open and closed loop mechanisms. It further avoids the known problems of choosing independent topological loops for a given closed loop mechanism. The absolute coordinate approach is reported in detail in this paper, and the relative coordinate formulation is shown for completeness.

The essential problem in characterizing the singular points of the forward kinematic map is that its differential is a map from the tangent space of the configuration space to $\mathbb{R}^3$. The novelty of our approach is to use the syzygy module of the differential of the constraint map to characterize these tangent spaces at different base points. This module can be represented as a matrix whose columns are the generators of the module and moreover these generators can actually be computed in pretty much the same way as the generators of a Gröbner basis of a given ideal. Composing this syzygy matrix with the differential of the forward kinematic map gives us a matrix whose Fitting ideals characterize the singular points of the forward kinematic map and give the desired stratification of the configuration space. Similar methods of computational algebraic geometry have been applied before for mechanisms and multibody systems with *closed kinematical loops* for example in [7–16]. The fact which makes our computations feasible also for more complicated mechanisms is the simplified joint constraints [13,17] given here in Appendix A. This is also explained in Section 3.2. An interesting alternative approach based on numerical homotopy continuation can be found for example in [18–20].

This same approach allows us to compute the boundary of the image of the forward kinematic map. This boundary is a subset of the image of the singular points, and the Zariski closure of this image can be computed. This in turn gives evidently a description of the image itself. The boundary of the workspace of regional manipulator has been previously investigated in [21] using manifold stratification where the boundary is obtained as an analytic variety. Whereas for analytic varieties the theory is usually *local*, in our approach the boundary is obtained as an algebraic variety for which the theorems are usually *global*.

The paper is organized as follows. In Section 2 we recall the necessary algebraic background. In Section 3 we present the needed constraints and the essential decompositions of the revolute joint constraints. In Section 4 we analyze our system using the absolute coordinates and in Section 5 we do the same thing using the relative coordinates. Finally in Section 6 we give some conclusions.

The actual computations were done with well established computer program Singular [22] for polynomial computations and the computations done in this article are almost completely automated.

2. Mathematical preliminaries

Here we quickly review the necessary tools that will be needed. For more details we refer to [23–29].

2.1. Ideals and varieties

Let us consider polynomials of variables $x_1,\dots,x_n$ with coefficients in the field $\mathbb{K}$ and let us denote the ring of all such polynomials by $\mathbb{A}=\mathbb{K}[x_1,\dots,x_n]$. In the kinematical analysis we are given a set of polynomials $f_1,\dots,f_k\in\mathbb{A}$ and the configuration space is its zero set. This zero set is called the *variety* corresponding to the system of polynomials and the goal is to analyze the geometry of this variety using algebraic properties of polynomials.

The given polynomials $f_1,\dots,f_k\in\mathbb{A}$ generate an *ideal*:

$$\mathcal{I}=\langle f_1,\dots,f_k\rangle=\{f\in\mathbb{A}\,|\,f=h_1f_1+\dots+h_kf_k,\ \text{where } h_i\in\mathbb{A}\}.$$

We say that the polynomials f_i are *generators* of $\mathcal{I}$ and as a set they are the *basis* of $\mathcal{I}$. The variety corresponding to an ideal is

$$\mathsf{V}(\mathcal{I})=\{a\in\mathbb{K}^n|f(a)=0\quad\forall f\in\mathcal{I}\}\subset\mathbb{K}^n.$$

The *radical* of $\mathcal{I}$ is

$$\sqrt{\mathcal{I}}=\{f\in\mathbb{A}|\exists m\geqslant 1\text{ such that } f^m\in\mathcal{I}\}.$$

An ideal $\mathcal{I}$ is a radical ideal if $\mathcal{I}=\sqrt{\mathcal{I}}$. Note that $\mathsf{V}(\mathcal{I})=\mathsf{V}(\sqrt{\mathcal{I}})$. An ideal $\mathcal{I}$ is *prime* if the following holds:

$$fg\in\mathcal{I}\quad\Rightarrow\quad f\in\mathcal{I}\text{ or } g\in\mathcal{I}.$$

Evidently a prime ideal is always a radical ideal. Given 2 ideals their *sum* is defined by

$$\mathcal{I}+\mathcal{J}=\{f+g\in\mathbb{A}\,|\,f\in\mathcal{I},\quad g\in\mathcal{J}\}.$$

The geometric meaning of the definition is that

$$\mathsf{V}(\mathcal{I}+\mathcal{J})=\mathsf{V}(\mathcal{I})\cap\mathsf{V}(\mathcal{J}).$$

The following facts are fundamental