Applied Mathematics and Computation 232 (2014) 944–954

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Laplace transforms for approximation of highly oscillatory Volterra integral equations of the first kind ☆

Shuhuang Xiang

Department of Applied Mathematics and Software, Central South University, Changsha, Hunan 410083, PR China

ARTICLE INFO

Keywords:
Asymptotic
Bessel function
Highly oscillatory Volterra integral equations
Laplace transform
Numerical solution

ABSTRACT

This paper focuses on Laplace and inverse Laplace transforms for approximation of Volterra integral equations of the first kind with highly oscillatory Bessel kernels, where the explicit formulae for the solution of the first kind integral equations are derived, from which the integral equations can also be efficiently calculated by the Clenshaw–Curtis–Filon-type methods. Furthermore, by applying the asymptotics of the solution, some simpler formulas for approximating the solution for large values of the parameters are deduced. Preliminary numerical results are presented based on the approximate formulae and the explicit formulae, which are compared with the convolution quadrature and numerical inverse Laplace transform methods. All these methods share that the costs the same independent of large values of frequencies.

1. Introduction

In this paper we are concerned with the numerical solution of the Volterra integral equation of the first kind with a highly oscillatory Bessel kernel

$$\int_0^x J_m(\omega(x-t))y(t)dt = f(x), \quad x \in [0,1], \tag{1.1}$$

where $y(x)$ is the unknown function whose value is to be determined in the interval $[0,1], f(x)$ a given sufficiently smooth function with $f(0)$=0, J_m the Bessel function of the first kind and of nonnegative integer order m and ω a large parameter.

Equations of this type can arise from a retarded potential integral equation after their application of the continuous Fourier transform, and its numerical solution has attracted much attention during the last few years (see, for instance, [6,7,10] and the references therein). One feature of the Volterra integral Eq. (1.1) is of particular note: when $\omega \gg 1$, the kernel function $J_m(\omega(x-t))$ would become highly oscillatory. This means that the standard collocation method and discontinuous Galerkin method may suffer from difficulty due to that the length scale associated with the space mesh is too small, which will induce a large scale and ill-conditioned linear system [6,7,31]. It is also difficult to analyze the error bounds since we know little about the asymptotics of the solution of (1.1) on ω.

☆ This paper is supported by the National Science Foundation of China (Grant No. 11071260).
E-mail address: xiangsh@mail.csu.edu.cn

http://dx.doi.org/10.1016/j.amc.2014.01.054
0096-3003/

The theoretical and numerical aspects of the Volterra integral equation of the first kind

$$\int_a^x K(x,t)y(t)dt = f(x), \quad x \in [a,b], \tag{1.2}$$

have been investigated extensively. Numerical approaches for (1.2) have been constructed by replacing the integral by a numerical quadrature formula (see, for example, [3,5,6,14,18,23]). However, they can not be applied to (1.1) since the kernel $J_m(\omega x)$ is highly oscillatory for large values of ω, therefore, the computation of the integral containing $J_m(\omega x)$ by standard quadrature methods is exceedingly difficult and the cost steeply increases with ω [16,17] (also see Fig. 1 in [15]). It has been done for the special case $m = 0$, where the unique solution can be written by Brunner et al. [7] as

$$y(x) = f'(x) + \omega \int_0^x \frac{J_1(\omega(t))}{t} f(x-t)dt, \quad x \in [0,1], \tag{1.3}$$

which can be efficiently computed by the Filon-type quadrature [4,26,31–33,35].

Laplace transform for $g : [0,+\infty) \to (-\infty,+\infty)$ denoted by

$$L[g(x)] = \widetilde{g}(p) = \int_0^\infty e^{-px} g(x)dx$$

and inverse Laplace transforms denoted by

$$\widetilde{L}[\widetilde{g}(p)] = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{px}\widetilde{g}(p)dp$$

are particularly well suited to the study of convolution problems. An elementary and detailed discussion of transforms may be found in [9,2,8,27].

If $f(x)$ at the right-hand side of (1.1) is well defined in $[0,+\infty)$, then by using Laplace transforms

$$L[J_v(\omega x)] = \frac{\omega^v}{\sqrt{p^2+\omega^2}\left(p+\sqrt{p^2+\omega^2}\right)^v}, \quad v > -1, \tag{1.4}$$

(see [2,27]), the solution of (1.1) can be represented by $L[y(x)] \cdot L[J_m(\omega x)] = L[f(x)]$ and

$$L[y(x)] = \frac{L[f(x)]\sqrt{p^2+\omega^2}\left(p+\sqrt{p^2+\omega^2}\right)^m}{\omega^m}. \tag{1.5}$$

Then $y(x)$ can be calculated by using numerical algorithms on inverse Laplace transforms if $L[f(x)]$ can be explicitly represented by some special functions. However, for general cases, $L[f(x)]$ is often unknown. Moreover, it is well-known that numerical inversion of Laplace transforms is a notoriously ill-conditioned process [11,25].

The convolution quadrature method [19–21] can also be applied to solve (1.1). The approximate solution y_k at $x = kh$ with $h = 1/N$ is computed from the following linear system

$$\sum_{j=1}^{k} w_{k-j} y_j = f(kh), \quad k = 1,2,\ldots,N, \tag{1.6}$$

where the convolution quadrature weights w_j are determined from their generating power series $\sum_{j=1}^{\infty} w_j z^j = F(\delta(z)/h)$. Here $F(p)$ is the Laplace transform of $f(x)$, and $\delta(z)$ is a given function such as $\delta(z) = 1 - z$ or $\delta(z) = (1-z)/(1+z)$ [21]. For Eq. (1.1), the computation of the weights w_j from the expansion $L[J_m(\delta(z)/h)]$ is very difficult especially for large values of n.

In this paper, we shall focus on efficient methods for the numerical solution of (1.1). In Section 2, we will derive explicit formulae for the solution of (1.1). Based on the formulae, in Section 3, we establish the asymptotics of the solution, which lead to simpler approximation of the solution for large values of ω. Applying the approximation formula and explicit formula, in Section 4, we will consider efficient algorithms for (1.1), and compare with numerical inverse Laplace algorithms and convolution methods.

2. The explicit formula for the solution of (1.1)

For the Volterra linear equation of the first kind (1.2), it is well-known that

Lemma 2.1 ([5,12,14,23,27]). *Assume that the functions $f(x)$ and $K(x,t)$ in (1.2) are continuous together with their first derivatives on $[a,b]$ and on $S = \{a \leqslant x \leqslant b, \ \ a \leqslant t \leqslant b\}$, respectively. If $K(x,x) \neq 0$ ($x \in [a,b]$) and $f(a) = 0$, then there exists a unique continuous solution $y(x)$ of Eq. (1.2).*

Furthermore, in the case the functions $K(x,t)$ and $f(x)$ are continuous and (1.2) *has a unique solution, then the right-hand side of* (1.2) *must satisfy the following conditions*