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Existence and stability of equilibrium solutions of a nonlinear heat equation

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ABSTRACT

The main purpose of this paper is to investigate the existence and stability of periodic and non-periodic equilibrium solutions related to the nonlinear heat equation:

$$u_t = u_{xx} + wu + u^3 + u^5. \tag{1}$$

The existence of periodic equilibriums with a fixed period L is deduced from the Theory of Jacobian Elliptical Functions and the Implicit Function Theorem. We show that these periodic equilibriums tend to the non-periodic positive equilibrium solution in the real line. Our stability/instability results are obtained trough the spectral study of the linear operator associated to the linearized stability problem as well as the study of a certain scalar quantity.

1. Introduction

Let $w \in \mathbb{R}$ be a real parameter and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth real function, the semilinear parabolic equation (NLHE):

$$u_t = u_{xx} + wu + f(u), \tag{2}$$

describes the conduction of heat in a stationary medium of dimension one, the value $u(x,t)$ represents the temperature at position x at the time t and the nonlinearity $wu + f(u)$ say us that the heat is produced by a family of chemical reactions. Nowadays, in the literature, there is a huge amount of work about Eq. (2) involving different problems: existence, uniqueness and continuous dependence of solutions [8], existence and nonexistence of finite time blow-up of solutions [6,7,9,11], existence and stability of periodic and non-periodic equilibrium solutions [12], structure and bifurcation of global attractors [5], between others.

In this paper, we study stability/instability of equilibrium solutions to the Eq. (2) when $f(x) = x^3 + x^5$. For this purpose, our approach uses some ideas from the theory of stability of periodic travelling wave solutions for nonlinear dispersive equations. Indeed, if we consider $f(u) = u^3$ in (2), from [1] it is possible to check that a periodic equilibrium solution associated to the equation

$$u_t = u_{xx} - wu + u^3 \tag{3}$$

is given by

$$u(x,t) = \phi_w(x) = \eta_1 \mathrm{dn}\left(\frac{\eta_1}{\sqrt{2}}x; k\right), \tag{4}$$

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this equilibrium solution is unstable, moreover, the *unstable manifold* associated to the equilibrium solution (4) has exactly dimension one. For a proof of the last affirmation, see Theorem 3.1 in [1]. Similarly, considering $f(u) = u^5$ in (2), analogous results on existence and stability of periodic equilibrium solutions for the equation

$$u_t = u_{xx} - wu + u^5,$$

can be deduced from [3]. Applications to other models, see [2].

For a better understanding of our work, it is divided in four sections. In the first section we establish some notations. The second section will be mainly dedicated to establish a general criterion that will allow us to conclude the instability of equilibrium solutions to the Eq. (2). The third section will be dedicated to calculate positive periodic and non-periodic equilibrium solutions to the Eq. (1). In the fourth section, we will establish our periodic and non-periodic stability/instability results.

First, we fix some notations and definitions.

1.1. Notations

- For $s \in \mathbb{R}$ and $L > 0, H^s_{per}([0,L])$ will be denote the Sobolev space of periodic functions of $L^2([0,L])$ type. In the same way, $H^s(\mathbb{R})$ will be denote the Sobolev space of functions of $L^2(\mathbb{R})$ type. We will use H^s to denote both $H^s(\mathbb{R})$ or $H^s_{per}([0,L])$. For definitions and properties of the spaces H^s, see [10].
- For $w < 0$, we will denote by $A : D(A) \to X := L^2_{per}([0,L])$ the linear operator defined as

$$A := -\partial_{xx} - w, \tag{5}$$

where $D(A) := H^2_{per}([0,L])$ denotes the domain of the operator A. It is well known that A is a positive self-adjoint operator with spectral set given by

$$\sigma(A) = \{\lambda_k := 4\pi^2k^2/L^2 - w | k \in \mathbb{Z}\},$$

where all of the λ_k are eigenvalues. The eigenspace associated to λ_0 is unidimensional and generated by the function $\Theta_0 \equiv 1/\sqrt{2}$. On the other hand, for $k > 0$, the eigenspace associated to the eigenvalue λ_k is bidimensional and it is generated by the set $\{\Theta_k, \Theta_{-k}\}$, where $\Theta_k := \cos(2k\pi x/L)$ and $\Theta_{-k} := \sin(2k\pi x/L)$. Using spectral theorem for self-adjoint operators we can represent the operator A in the following way

$$Ag = \sum_{k=-\infty}^{\infty} \lambda_k \langle g, \Theta_k \rangle \Theta_k.$$

- For $\alpha \in \mathbb{R}$, and the operators $A^\alpha : D(A^\alpha) \to L^2_{per}([0,L])$ given by

$$A^\alpha g = \sum_{k=-\infty}^{\infty} \lambda_k^\alpha \langle g, \Theta_k \rangle \Theta_k,$$

with, $D(A^\alpha) := L^2_{per}([0,L])$ for $\alpha \le 0$ and $D(A^\alpha) := R(A^{-\alpha})$ for $\alpha > 0$. We define the Banach spaces X_α as:

$$X_\alpha := D(A^\alpha), \ \ ||g||_\alpha := ||A^\alpha g||_{L^2_{per}}.$$

It is not difficult to see that X_α is isomorphic to $H^{2\alpha}_{per}[0,L]$.

2. Instability criterion

Now, we are going to establish a general framework to deal with instability of equilibrium solutions to the Eq. (2). We begin by recalling that an *equilibrium solution* to the NLHE is a solution to the Eq. (2) in the form:

$$u(x,t) = a(x). \tag{6}$$

Therefore, substituting (6) in (2), we obtain that a have to satisfy the following nonlinear ordinary differential equation:

$$a'' + wa + f(a) = 0. \tag{7}$$

If, for instance, $f(0) = 0$, then clearly for $w \in \mathbb{R}$ the null solution

$$a_w \equiv 0 \tag{8}$$

is a family of equilibrium solutions to the Eq. (2).

Now, we will do the following assumptions:

- **(H-1)** We will suppose the existence of a curve

$$w \in \Omega \to a_w \in H^1,$$

of nontrivial equilibrium solutions to the nonlinear heat equation (2). Here Ω denotes an open set of real numbers.