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A Combining Method for solution of nonlinear boundary value problems

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ABSTRACT

Modeling-based simulation techniques and numerical methods such as Differential Quadrature Method and Integral Quadrature Method are widely used for solution of ordinary differential equations. However simulation techniques do not allow to impose boundary conditions, and both Differential Quadrature Method and Integral Quadrature Method have some deficiencies in applying multiple boundary conditions at the same location. Moreover they are not convenient for the solution of non-linear Ordinary Differential Equations without using any linearization process such as Newton–Raphson technique and Frechet derivative which requires an iterative procedure increasing the time needed for solution. In this study, modeling-based simulation technique is combined with Differential Quadrature Method and/or Integral Quadrature Method to eliminate the aforementioned deficiencies. The proposed method is applied to four different nonlinear boundary value problems including a coupled nonlinear system, second and fourth order nonlinear boundary value problems and a stiff nonlinear ordinary differential equation. The numerical results obtained using Combining Method are compared with existing exact results and/or results of other methods. Comparison of the results show the potential of Combining Method for solution of nonlinear boundary value problems with high efficiency and accuracy, and less computational work.

1. Introduction

In seeking a more efficient method using just a few grid points to obtain accurate numerical results, the Differential Quadrature Method (DQM) was proposed by Bellman and Casti [1], Bellman et al. [2]. Bellman et al. [2] suggested two approaches to determine the weighting coefficients of the first order derivative. In the first approach, it is very difficult to obtain the weighting coefficients for a large number of grid points. The second one uses a simple algebraic formulation, but with the coordinates of grid points chosen as the roots of the shifted Legendre polynomial. In order to deal with these drawbacks, Quan and Chang [3] used Lagrange interpolation polynomials as test functions, and then obtained explicit formulations to determine the weighting coefficients of the first and second order derivatives. Later, Shu and Richards [4] derived an identical formula for the weighting coefficients of the first order derivative and provided a recurrence relationship which can generate weighting coefficients for any second and high order derivatives from the first order derivative weighting coefficients. The method is so called as Generalized Differential Quadrature Method (GDQM).

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Nevertheless, many problems encountered in mechanics are governed by higher order differential equations with more than one boundary condition at the same point. A δ-point approximation approach was first proposed by Jang et al. [5] to apply the double boundary conditions in fourth order systems. The δ-points are chosen at a small distance $\delta \cong 10^{-5}$ adjacent to the boundary points. Then, two boundary conditions are applied at the boundary point itself and its adjacent δ-point. But there are some shortcomings of δ-technique. One boundary condition is exactly satisfied at the boundary while the other is approximately satisfied at the δ-point. To obtain an accurate solution, the δshould be chosen to be very small. Usage of too small values for δ may cause ill conditioned weighting coefficient matrices and unexpected oscillation behavior of the solutions. In order to overcome the drawbacks of the δ-technique, Wang and Bert [6] presented a new technique of incorporating the boundary conditions in the weighting coefficient matrices in the context of plate vibration problems. However, there are some limitations to the application of this approach. The method cannot be used for clamped–clamped and free boundaries of plates [6].

The Generalized Differential Quadrature Rule (GDQR) was proposed recently to solve differential equations which involve more than one boundary or initial conditions at the same point [7,8]. The GDQR has been used to solve nonlinear initial value differential equations of second and fourth orders [9]. The nonlinearity is dealt with using the Frechet derivative to convert the nonlinear differential equation into linear differential equation and an iterative procedure is used to solve the linear differential equation. However, the iterative technique in the solution process has increased the computation time. Another drawback of this method is usage of higher order polynomials which increases the computational efficiency.

The original Combining Method (CM) was proposed by Girgin [10] especially for solving nonlinear differential equations. Girgin point out that the CM can be easily applied to solve high order nonlinear ODEs, which may have multiple boundary conditions at the same point without using any special technique. But, this method is not useful if pure function value is not present in the equation. Consequently, Girgin proposed Modified Integral Quadrature Method [11] by extending the Generalized Integral Quadrature Method (GIQM) which is developed by Shu et al. [12]. This method is different from GIQM in a way that one can obtain function values precisely from its differential form by using integral constant matrix.

In this study, a new Combining Method is proposed in which both weighting coefficients of derivatives and integrals are employed in the solution process and applied to solution of nonlinear boundary value problems. Especially, coupled nonlinear ODE systems and stiff nonlinear ODEs are considered. Multiple boundary conditions are imposed at the same point and drawbacks of δ-technique and other techniques to implement two or more boundary conditions at the same location are eliminated. Moreover, the method does not need any use of linearization technique.

2. Calculation of weighting coefficients

Shu and Richards [4], introduced a simple algebraic formulation to compute the weighting coefficients of the first order derivative without any restriction on the choice of the grid points, and a recurrence relationship to compute the weighting coefficients of the second and higher order derivatives. They used Lagrange interpolation shape function as trial function to obtain the explicit weighting coefficients. This method will be used in the present study to obtain weighting coefficients of derivatives for an easy and efficient implementation of the CM. In order to obtain function values from its differential form, Modified Integral Quadrature Method is used and weighting coefficients of integrals are obtained.

2.1. Weighting coefficients of Differential Quadrature

Consider a function $f(x)$ prescribed in a field domain $a \leqslant x \leqslant b$. Let $f(x_i)$ be the function values specified in a finite set of N discrete points $x_i (i = 1, 2, \ldots, N)$ of the field domain. The rth-order derivative of $f(x)$ at any discrete point x_i can be written in DQ analog form as:

$$\frac{d^r f(x_i)}{dx^r} = \sum_{j=1}^{N} a_{ij}^{(r)} f(x_j), \quad i = 1, 2, \ldots, N \tag{1}$$

where $a_{ij}^{(r)}$ are the weighting coefficients of the rth-order derivative of the function $f(x)$ associated with points x_i.

In the GDQ, the test functions are assumed to be the Lagrangian interpolation test functions such as

$$\ell_j(x) = \frac{\phi(x)}{(x - x_j)\phi^{(1)}(x_j)}, \quad j = 1, 2, \ldots, N \tag{2}$$

where

$$\phi(x) = \prod_{m=1}^{N} (x - x_m); \quad \phi^{(1)}(x_j) = \frac{d\phi(x_j)}{dx} = \prod_{m=1, m\neq j}^{N} (x_j - x_m). \tag{3}$$

The weighting coefficients of the first-order derivative can be obtained using Lagrange interpolation polynomial as follows:

$$a_{ij}^{(1)} = \frac{d\ell_j(x_i)}{dx} = \frac{\phi^{(1)}(x_i)}{(x_i - x_j)\phi^{(1)}(x_j)}, \quad i,j = 1, 2, \ldots, N, \quad i \neq j \tag{4}$$