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Exponential higher-order compact scheme for 3D steady convection–diffusion problem

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ABSTRACT

In this paper, the exponential higher order compact (EHOC) finite difference schemes proposed by Tian and Dai (2007) [10] for solving the one and two dimensional steady convection diffusion equations with constant or variable convection coefficients are extended to the three dimensional case. The proposed EHOC scheme has the feature that it provides very accurate solution (exact in case of constant convection coefficient in 1D) of the homogeneous equation that is responsible for the fundamental singularity of the homogeneous solution while approximates the particular part of the solution by fourth order accuracy over the nineteen point compact stencil. The key properties of this scheme are its stability, accuracy and efficiency so that high gradients near the boundary layers can be effectively resolved even on coarse uniform meshes. To validate the present EHOC method, three test problems, mostly with boundary or internal layers are solved. Comparisons are made between numerical results for the present EHOC scheme and other available methods in the literature.

1. Introduction

Convection–diffusion equation plays an important role in computational fluid dynamics (CFD) to simulate flow problems. Therefore, accurate and stable difference representations of the convection–diffusion equations are of vital importance. It has been known that the classical numerical methods on uniform grids give stable numerical solutions for singularly perturbed convection–diffusion problems only if one uses a very large number of grid points. As a result, upwind techniques have been proposed for solving convection–diffusion equations. However, they are less accurate and may not capture the sharp gradients in the solution. In the last two decades, higher order compact (HOC) finite difference (FD) schemes, which are computationally efficient, were developed. There are two types of higher order compact schemes, namely, HOC polynomial and HOC exponential schemes. Several authors developed a number of HOC polynomial FD schemes for convection–diffusion equations on uniform grids for two dimensional spaces [1–3] and three dimensional spaces [4–6]. However, the existing HOC polynomial FD schemes are not suitable for particular physical problems, such as abrupt boundary layer in convection-dominated problems, unless a very fine mesh is used. This dilemma can be resolved by utilizing nonuniform mesh or local mesh refinement strategies. Polynomial HOC on nonuniform grid have been developed by Kalita et al. [7] for 2D steady convection–diffusion equation and by Ge et al. [8] for solving 3D Poisson equation. However, in case of using nonuniform grids, the boundary layer location or the singularity region must be known before the construction of the non-uniform grid.

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In contrast, the exponential HOC (EHOC) scheme has the noteworthy feature that it provides very accurate solution for singular perturbation problems characterized by boundary and/or transition layers where the gradients of the solution are large. Pillai [9] and Tian and Dai [10] developed a fourth order compact (4OC) exponential FD scheme for solving 1D and 2D convection–diffusion equations with constant and variable convection coefficients on compact stencil. Mishra and Yedida [11,12] developed a 6OC exponential FD scheme for solving 1D convection–diffusion equations with constant convection coefficients and a 4OC exponential scheme for 2D equations.

The main aim of this work is to extend Tian and Dai's work on EHOC schemes for solving 1D and 2D steady convection diffusion equations to the 3D. This EHOC scheme produces fourth order accuracy even with variable convection coefficients and source term. We start with the 1D case and prove that EHOC scheme produces exact solution in case of constant convection coefficients and constant source term. Based on the EHOC scheme proposed for 1D, we derive the EHOC scheme for 3D convection–diffusion equation with variable coefficients. The numerical results of the present EHOC scheme are compared with the numerical solutions of polynomial HOC [4]. The numerical results exhibit that the scheme proposed in the present work agrees very closely with the exact solution and resolves efficiently the high gradients near the boundary layer areas without refining the mesh.

The organization of this paper is as follows. Section 2 presents an EHOC scheme for 1D steady convection–diffusion equations. Brief extension to the 2D case is reviewed in Section 3. Section 4 presents the derivation details of an EHOC scheme for 3D steady convection–diffusion equation with constant convection coefficients. Then the EHOC scheme for variable convection coefficients is developed. Section 5 presents some numerical experiments to validate theoretical remarks and demonstrates the effectiveness of the present method. Finally we draw conclusions in Section 6.

2. High-order compact exponential FD methods: 1D case

In this section we consider the steady one-dimensional non homogeneous convection–diffusion model problem

$$-au_{xx} + d(x)u_x = f(x) \tag{1}$$

where the diffusion coefficient a is constant while the convection coefficient d and the source term f are sufficiently smooth functions with respect to x. It is well known that Eq. (1) is a linear differential equation whose solution $u(x) = u_h(x) + u_p(x)$ composes of two parts. The homogeneous part $u_h(x)$ is the general solution of the homogeneous equation $-au_{xx} + d(x)u_x = 0$ and is independent of the source function f. The second part is a particular solution $u_p(x)$ that satisfies Eq. (1).

Let the domain of the problem $0 \leqslant x \leqslant 1$ be divided uniformly into n intervals with mesh size h. The second order finite difference of Eq. (1) is

$$-aD_h^2u_i + d_iD_hu_i = f_i \tag{2}$$

where $D_hu_i = \frac{u_{i+1}-u_{i-1}}{2h}$, $D_h^2u_i = \frac{u_{i+1}+u_{i-1}-2u_i}{h^2}$ are the central difference approximations for the first and second derivatives, respectively and $u_i = u(x_i)$, $x_i = ih$, $i = 0, 1, \ldots, n$. Although scheme defined by Eq. (2) is second order, for convection dominant case ($a/d_i \ll 1$), the solution is very inaccurate since the scheme itself creates oscillatory solution. To avoid this, a small enough mesh size h must be used. Another alternative is the application of upwind schemes. A notable disadvantage of upwind schemes is the low order of approximation that they yield [[13], Section 2.2.4]. In a previous work [14], the oscillatory performance of Galerkin method for convection diffusion problems at high Peclet numbers was analyzed and a modified diffusion coefficient (MDC) technique that produces exact nodal solution was proposed for a class of 1D problems. Extending the MDC for other classes of 2D convection diffusion equations, non oscillatory solution was obtained but only with second order accuracy. Our main interest in this paper is to extend MDC to obtain high accuracy solution for 3D convection diffusion equation with variable coefficients by making use of some ideas from EHOC schemes. In the last few years EHOC schemes were proposed to solve convection dominant convection–diffusion equations efficiently in 2D ([10,12]). In this section we try to analyze and understand the basics of MDC and EHOC techniques and hence interpret their efficiency (and sometimes exactness) of solutions for convection–diffusion equations with boundary layers even on uniform grids.

2.1. Case 1: constant convection coefficient and constant right hand side

Consider the simple convection–diffusion equation with constant d and f.

$$-au_{xx} + du_x = f \tag{3}$$

The second order finite difference of Eq. (3) is

$$-aD_h^2u_i + dD_hu_i = f \tag{4}$$

For $d \neq 0$. It is easy to show that Eq. (3) has the exact solution

$$u(x) = c_1e^{\frac{dx}{a}} + c_2 + \frac{f}{d}x \tag{5}$$

where c_1 and c_2 are arbitrary constants determined by the boundary conditions. The homogeneous solution is $u_h(x) = c_1e^{\frac{dx}{a}} + c_2$ while the particular solution is $u_p(x) = \frac{f}{d}x$. It is important to note that, for convection dominated problems