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Exact multiplicity and stability of solutions of second-order Neumann boundary value problem

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ABSTRACT

The purpose of this paper is to establish the exact multiplicity and stability of solutions of the equation $u'' + g(x,u) = f(x)$ with the Neumann boundary value conditions $u'(0) = u'(1) = 0$. Exactly three ordered solutions are obtained by taking advantage of the anti-maximum principle combined with the methods of upper and lower solutions. Moreover, we obtain that one of three solutions is negative, while the other two are positive, the middle solution is unstable, and the remaining two are stable.

1. Introduction

In this paper, we consider the exact multiplicity and stability of solutions for the Neumann boundary value problem

$$\begin{cases} u'' + g(x,u) = f(x), & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f(x)$ is a continuous function and $g(x,u)$ is continuous and locally differentiable with respect to the second variable with

$$g'(x,u) \ll \pi^2/4, \quad \text{for all } x \in [0,1],$$

namely, $g'(x,u) \leqslant \pi^2/4$, with the strict inequality on a set of positive measure.

The Neumann problem has played a significant role in mathematical physics (for example, equilibrium problems concerning beams, columns, or strings and so on), and hence has attracted the attention of many researchers over the last two decades. The existence and multiplicity of positive solutions for Neumann boundary value problem were investigated by the fixed point theorems in [1,2]. Most of the results on the Neumann problem are about the unique solution, or the least number of solutions in the previous literature. Relatively few studies have been written about the exact multiplicity of solutions for (1.1). The method of lower and upper solutions coupled with the monotone iterative has been applied successfully to obtain existence and approximation of solutions for Neumann boundary value problems classically when the lower solution is under the upper solution (see [3,4] and the references therein). The nonclassical case, i.e. the lower solution is over the upper solution, has been treated in many papers, for instance in [5] the periodic case was considered, in [6] the Neumann problem was studied. If the lower solution is over the upper solution, the monotone method is not valid in general. Anti-maximum principle plays an important role in applying the method of lower and upper solutions, see [7] and references

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therein. In [8], the exact multiplicity and stability of periodic solutions of the Duffing equation with convex nonlinearity were studied. More recently, in [9–11], the exact multiplicity and stability of periodic solutions under the effects of concave-convex nonlinearity were investigated. In [12], the exact multiplicity of solutions was studied for the semipositone problems with concave-convex nonlinearity. In this paper, the concavity and convexity of the nonlinearity are not required but only the monotonicity of the nonlinearity is demanded. We obtain the stability and exact number of the positive and negative solutions for the Neumann boundary value problem (1.1).

The organization of this paper is as follows: we shall introduce the main results in the rest of this section. In Section 2, we present some useful preliminaries. Then in Section 3, the proof of the main results are given. In Section 4, one example is presented to illustrate the main results.

Theorem 1.1. *Let $f(x) > 0$. Assume that the function $g(x,u)$ satisfies the following conditions:*

(i) *there exists $a < 0 < b < c$ such that $g(x,a) = g(x,0) = g(x,c) = 0$, $\max_{u\in(0,c)} g(x,u) = g(x,b) = M$ for all $x \in [0,1]$, $\lim_{u\to-\infty} g(x,u) = Q(x)$, uniformly in $x \in [0,1]$ with $Q(x) > M > 0$;*
(ii) *$g(x,u) > 0$ for $u \in (-\infty,a) \cup (0,c)$ and $g(x,u) < 0$ for $u \in (a,0)$ for all $x \in [0,1]$;*
(iii) *$g'(x,u) < 0$ for $u \in (-\infty,a) \cup (b,c)$ and $0 < g'(x,u) \ll \pi^2/4$ for $u \in [0,b]$ and all $x \in [0,1]$.*

Then

(1) *(1.1) has no solution if $f(x) > Q(x)$ for all $x \in [0,1]$;*
(2) *(1.1) has a unique solution that is negative and stable if $Q(x) > f(x) > M$ for all $x \in [0,1]$;*
(3) *(1.1) has exactly three ordered solutions if $0 < f(x) < M$. Moreover, the minimal solution is negative and the other two are positive; also, the middle solution is unstable and the remaining two are stable.*

2. Preliminaries

Firstly, some notations are introduced for later use. Let $\mu_1(s(x)) < \mu_2(s(x)) \leqslant \mu_3(s(x)) \leqslant \cdots$ all be eigenvalues of the equation

$$\begin{cases} u'' + s(x)u + \mu u = 0, & x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{2.1}$$

It is well known that the first eigenvalue $\mu_1(s(x))$ is simple and the corresponding eigenfunction does not change sign. When $s(x) \equiv 0$, the first eigenvalue is equal to 0 and the second eigenvalue is equal to π^2.

The stability of the solutions is also an important subject in our study.

Definition 2.1. [13]. Suppose that u is the solution of (1.1). Then u is stable if the principal eigenvalue $\mu_1(g'(x,u))$ of the equation

$$\begin{cases} \varphi'' + g'(x,u)\varphi = -\mu\varphi, & x \in (0,1), \\ \varphi'(0) = \varphi'(1) = 0, \end{cases}$$

is strictly positive. The solution u is unstable if the principal eigenvalue $\mu_1(g'(x,u))$ is negative.

In order to apply the method of lower and upper solutions, the following anti-maximum principle is essential, which is one of the important tools in this paper. Next we consider the nonhomogeneous differential equation

$$\begin{cases} u'' + s(x)u = f(x), & x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{2.2}$$

Lemma 2.1. *Let $f(x) > 0$ on $[0,1]$ and $s(x)$ satisfy*

$$s(x) \ll \pi^2/4.$$

If $u(x)$ is a solution of (2.2), then the following statements hold:

(1) *either $u(x) > 0$ or $u(x) < 0$ for all $x \in [0,1]$;*
(2) *$u(x) > 0$ for all $x \in [0,1]$, if $\mu_1(s(x)) < 0$;*
(3) *$u(x) < 0$ for all $x \in [0,1]$, if $\mu_1(s(x)) > 0$.*