Applied Mathematics and Computation 230 (2014) 1–8

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Global behavior of the higher order rational Riccati difference equation

Azizi Raouf

University of Carthage, Faculty of Sciences of Bizerte, Department of Mathematics, 7021 Zarzouna, Tunisia

ARTICLE INFO

Keywords:
Higher order Riccati difference equation
Second order
Forbidden set
Periodic solutions
Asymptotic stability
Dense solutions

ABSTRACT

Let k be a positive integer and $a_0, a_1, \ldots, a_k$ be non-negative real numbers with $a_k > 0$. We show that if $\gcd\{i;\ a_{i-1} > 0,\ 1 \leqslant i \leqslant k+1\} = 1$ then the rational Riccati difference equation of order k

$$x_{n+1} = a_0 + \frac{a_1}{x_n} + \frac{a_2}{x_n x_{n-1}} + \cdots + \frac{a_k}{x_n x_{n-1} \cdots x_{n-k+1}}, \quad n = 0, 1, 2, \ldots$$

has a unique positive equilibrium point that is stable and attracts all solutions with initial points outside a set of zero Lebesgue measure. This holds in particular if $a_0 + a_{k-1} > 0$. The case $k = 3$ is studied in detail.

1. Introduction

We consider the rational Riccati difference equation of order k ($k \geqslant 1$) defined by:

$$x_{n+1} = a_0 + \frac{a_1}{x_n} + \frac{a_2}{x_n x_{n-1}} + \cdots + \frac{a_k}{x_n x_{n-1} \cdots x_{n-k+1}}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

with initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0 \in \mathbb{R}$, and the parameters $a_0, a_1, \ldots, a_k$ are real numbers with $a_k \neq 0$.

If $k = 1$ then Eq. (1.1) is reduced to the first order Riccati rational equation which has been studied thoroughly (see, for example, [1,3,5,6]). If $k = 2$, Dehghan et al. [3] investigated Eq. (1.1) under the conditions $a_0 \geqslant 0, a_1 \geqslant 0, a_0 + a_1 > 0, a_2 > 0$. They proved that the equation, in the stated range of parameters, had a positive fixed point that is stable and attracts all solutions having initial points outside a plane set of zero Lebesgue measure. The case $k = 2$ with arbitrary real parameters a_0, a_1, a_2 is studied by the author (see [1]). In the present paper, we study Eq. (1.1) for any order k and non-negative parameters. i.e. $a_0, a_1, \ldots, a_{k-1} \geqslant 0$ and $a_k > 0$. In fact we prove a similar result to that in [3] concerning the existence of fixed solutions and their stability. So far, there have been no results for $k \geqslant 3$. It was only Sedaghat [7] who tried to study this equation for $k = 3$ under specific conditions on parameters.

This paper is organized as follows: In Section 2, we transform Eq. (1.1) into a linear homogeneous difference equation $y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k}$ through a change of variable $x_n = \frac{y_n}{y_{n-1}}$. The yielding linear equation is used to obtain a representation of solution of Eq. (1.1) and then to determine the forbidden set of Eq. (1.1). For some results on these sets see, for example, [1,3,6,8]. In Section 3, we describe the asymptotic behavior and stability proprieties of the solutions under the condition $\gcd\{i;\ a_{i-1} > 0,\ 1 \leqslant i \leqslant k+1\} = 1$. Finally, Section 4 is devoted to the case $k = 3$, where we give a full study of the solutions of Eq. (1.1).

E-mail address: araoufazizi@gmail.com

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.12.055

2. The forbidden set

Similarly to the first and second order Riccati difference equations, Eq. (1.1) can be transformed into a linear difference equation

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k}. \tag{2.1}$$

of order $k+1$ through a change of variable $x_n = \frac{y_n}{y_{n-1}}$. If we define the initial values for Eq. (2.1) as $y_{-k} = 1$ (or any fixed non-zero real number), $y_{-k+1} = x_{-k+1}, y_{-k+2} = x_{-k+2}x_{-k+1}, \ldots, \; y_0 = x_0 x_{-1} \cdots x_{-k+1}$, then we obtain a one-to-one correspondence between the solutions of Eq. (1.1) and those solutions of Eq. (2.1) that do not pass through the origin. The characteristic polynomial of Eq. (2.1) is

$$P(X) := X^{k+1} - a_0 X^k - a_1 X^{k-1} - \cdots - a_k.$$

By basic linear theory, Eq. (2.1) can be solved explicitly and the solution $\{y_n\}_{n \geqslant -k}$ is expressed in terms of roots of P. The following proposition gives the general form of solution of Eq. (2.1) in terms of initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0$.

Proposition 2.1. *Let $\{y_n(x_{-k+1}, x_{-k+2}, \ldots, x_0)\}_{n \geqslant -k}$ be the solution of Eq. (2.1) with initial values*

$$y_{-k} = 1, \quad y_{-k+1} = x_{-k+1}, \quad y_{-k+2} = x_{-k+2}x_{-k+1}, \ldots, \quad y_0 = x_0 x_{-1} \cdots x_{-k+1}.$$

Then

$$y_n = \sum_{i=0}^{k-1} \beta_{i,n} x_{-i} x_{-i-1} \cdots x_{-k+1} + \beta_{k,n},$$

where $\beta_{i,n} = \beta_{i,n}(a_0, a_1, \ldots, a_k), \; 0 \leqslant i \leqslant n$.

Proof. We let $Y_n = \begin{pmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}$. We have

$$Y_{n+1} = \begin{pmatrix} y_{n+1} \\ y_n \\ \vdots \\ y_{n-k+1} \end{pmatrix} = \begin{pmatrix} a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k} \\ y_n \\ \vdots \\ y_{n-k+1} \end{pmatrix} = A Y_n,$$

where

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Hence $Y_n = A^n Y_0 = A^n \begin{pmatrix} x_0 x_{-1} \cdots x_{-k+1} \\ x_{-1} x_{-2} \cdots x_{-k+1} \\ \vdots \\ x_{-k+1} \\ 1 \end{pmatrix}$. Now set $A^n = \begin{pmatrix} \beta_{0,n} & \beta_{1,n} & \cdots & \beta_{k,n} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$. It follows that

$$y_n = \sum_{i=0}^{k-1} \beta_{i,n} x_{-i} x_{-i-1} \cdots x_{-k+1} + \beta_{k,n}. \qquad \square$$

The forbidden set of Eq. (1.1) can be written as

$$\mathcal{F} = \bigcup_{n \geqslant -k} \{(x_{-k+1}, x_{-k+2}, \ldots, x_0) \in \mathbb{R}^k; \; y_n(x_{-k+1}, x_{-k+2}, \ldots, x_0) = 0\}.$$

The next result is an immediate consequence of Proposition 2.1.

Proposition 2.2. *The forbidden set of Eq. (1.1) is given by*

$$\mathcal{F} = \bigcup_{n \geqslant -k} \left\{ (x_{-k+1}, \ldots, x_0) \in \mathbb{R}^k; \; \sum_{i=0}^{k-1} \beta_{i,n} x_{-i} x_{-i-1} \cdots x_{-k+1} + \beta_{k,n} = 0 \right\},$$