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Quadratic model of inter-population interaction: Investigation of stability areas

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ABSTRACT

A development quadratic model of the heterogeneous biological population consisting of a few sub-populations is investigated. The classic stabilization problem of solutions for the system of ordinary quadratic differential equations, which describing this model, is solved. Examples are given.

1. Introduction

The logistic equation

$$\dot{x}(t) = ax(t)\left(1 - \frac{x(t)}{k}\right), \quad x(t) \geqslant 0,\ a > 0,\ k > 0 \tag{1}$$

is a very classical (but still actual) tool for description of population growth [1]. The equation is based on the use of two fundamental population characteristics: a is the coefficient of productivity, which characterizes potential intensity of population's growth, and k is the"environment volume". This coefficient reflects an equilibrium size of the population, which can be realized in considering environment. Naturally, the description of a population by two positive real numbers is an essential simplification. Nevertheless this simplification is quite useful and popular [2,3].

One of factors, which are not taken into account in the model (1), is heterogeneity of populations. It is clear that different specimens have different abilities, both of growth rate and potential density. In (1) one considers average values; it is quite correct if there is some normal distribution of the indices. The real situation is usually more complex: a population can be divided for some more or less discrete groups (sub-populations) and indices within each of them are more similar each other that between the groups. Forming the population these groups are not completely isolated; it is very important that they can "exchange" by specimens (both directly and through reproduction).

Each ith subpopulation is still characterized by one value k_i ($i = 1, \ldots, n; n$ is a number of the groups) which means stable size of the subpopulation in the environment (in the case of absence of other sub-populations!). In the same time the growth coefficient should reflect possibility of production by this subpopulation of new members for other sub-populations. One should consider a number of a_{ij} ($i, j = 1, \ldots, n$), which reflects an intensity of production of specimens of jth subpopulation by ith one ($i \neq j$). The coefficient $a_{ii}; i = 1, \ldots, n$, describes the growth ("self-support") of ith subpopulation. It is quite logical to propose for the description of general dynamics of the heterogeneous biological population the following system equations:

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$$\begin{cases} \dot{x}_1(t) = \sum_{j=1}^{n} a_{1j}x_j(t)\left(1 - \frac{x_j(t)}{k_j}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ \dot{x}_n(t) = \sum_{j=1}^{n} a_{nj}x_j(t)\left(1 - \frac{x_j(t)}{k_j}\right). \end{cases} \quad (2)$$

Here $x_i(t)$ is a vector of states; the vector of initial values $\mathbf{x}^T(0) = (x_{10}, \dots, x_{n0})$ is given; a_{ij} and $k_j > 0$ are real numbers; $i,j = 1, \dots, n$.

Nature of the sub-populations can be quite different. They can be real sub-populations with genetic peculiarities and essentially isolated geographically (and in this case the diagonal coefficients a_{ii} will be much bigger then other ones). System (2) can also describe intensive interaction between representatives of different genetic lines, which can take place as a permanent component of population dynamics [4]. As example of such process one can mention interaction between human races or a change of quantities of blondes and brunettes in the human population. The model can be applied for any kind of heterogeneity, but in some case (for example, for age heterogeneity) it is transformed to very special degenerate kind.

Another possibility it is't consider genetic, but "social" groups. An existence of "horizontal" hierarchy (a separation of populations for families, bevies and so on) means the existence of "vertical" one (there are "social classes" of specimens: ordinary specimens, heads of family, bevy leaders) [5]. Contrary to sex difference specimens can change their group membership, although this change is not as mechanistic as change of age. Big populations of "intellectual" animals (particularly primates, ungulates, cetaceans, and even insects and fishes [6,7]) often form multi-level social structures. There are quite fundamental descriptions of social structures of several special species [8,9]. In some cases (for example, for Galapagos sea-lion [9]) it was even possible to estimate the number of layers in population social structure. Number of the levels depends on their both intellectual and "energetic" abilities [5].

Although model (2) has quite evident form and mentioned in scientific literature [10–12], it is not deeply discussed and investigated. Partially, it can be explained by quit high level of its mathematical complexity and big number of its parameters. Some first steps of investigation of the model was done in paper [13], where model (2) was called as "zygote elimination model" (several group of cells were considered as sub-populations; it is another opportunity to interpret the model). In current article we want to continue the investigation on quite formal mathematical level.

Let $x_{10} > 0, \dots, x_{n0} > 0$. Then generally speaking there exists a finite time t^* and $i \in \{1, \dots, n\}$ such that $x_i(t^*) < 0$. If the last inequality takes place, then from the biological point of view model (2) is improper. Nevertheless if $\forall t > 0$, we have $x_i(t) > 0; i = 1, \dots, n$, then model (2) can correctly describe the dynamic behavior of some heterogeneous biological population.

In [14–16] the existence conditions of the asymptotic stability cone for general quadratic systems were resulted. However these conditions have a local character. A main task of the present paper is constructing of a global stability region for system (2).

2. Preliminary results

In this section we remind some known results which are resulted in works [14–16].

Consider the homogeneous quadratic system of order n:

$$\begin{cases} \dot{x}_1(t) = \mathbf{x}^T(t)B_1\mathbf{x}(t), \\ \dots\dots\dots\dots\dots, \\ \dot{x}_n(t) = \mathbf{x}^T(t)B_n\mathbf{x}(t) \end{cases} \quad (3)$$

with the vector of initial values $\mathbf{x}^T(0)$. Matrices $B_1, \dots, B_n \in \mathbb{R}^{n\times n}$ are real and symmetrical.

Notice that any quadratic form in the right-hand side of system (3) can be written as

$$\mathbf{x}^T B_i \mathbf{x} = (\mathbf{r}_{i1}, \dots, \mathbf{r}_{in}) \cdot (x_1\mathbf{x}^T, \dots, x_n\mathbf{x}^T)^T,$$

where $\mathbf{r}_{i1}, \dots, \mathbf{r}_{in} \in \mathbb{R}^n$ are row-vectors of the matrix $B_i, i = 1, \dots, n$. Thus, any system (3) can be presented by the such form

$$\dot{\mathbf{x}}(t) = \mathbf{T} \cdot (\mathbf{x}(t) \otimes \mathbf{x}(t)), \quad (4)$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{r}_{11}, & \dots & , \mathbf{r}_{1n} \\ \vdots & \dots & \vdots \\ \mathbf{r}_{n1}, & \dots & , \mathbf{r}_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n^2}$$

and $\mathbf{T}$ is a mixed tensor of rank 3 (once contravalent and twice covalent); $\mathbf{x} \otimes \mathbf{x} = (x_1\mathbf{x}^T, \dots, x_n\mathbf{x}^T)^T$ is a tensor product of the vector $\mathbf{x}$ with itself. (Here the tensor $\mathbf{T}$ is realized as an element of the space matrices of sizes $n \times n^2$.)