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Arclength numerical continuation in free-boundary flow

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ABSTRACT

Using Helmholtz's wake model, we reduce the study of the free boundary flow past an obstacle consisting of an arc of circle to the investigation of a Hammerstein nonlinear integral equation depending on a real parameter λ. The papers dedicated to this problem investigated the case $\lambda > 0$ which corresponds to a convex obstacle with respect to the incoming fluid. Herein, we apply for the first time in the literature the arclength continuation method for the case $\lambda < 0$ corresponding to a concave arc of a circle. For $\lambda > 0$ the existence and the uniqueness of the solution was demonstrated, but for $\lambda < 0$, depending on its value compared to the one of a turning point, the integral equation has either no solution or two distinct solutions corresponding to two different obstacles. We numerically calculate the free lines, the velocity field and the stream lines. A diagram of the drag coefficient versus the arc measure for both convex and concave obstacles suggests us to draw some conclusions concerning the optimization of the blades of a vertical axis (Savonius) wind turbine.

1. Introduction

Usually, in papers dealing with the classical theory of the 2*D* potential flow of an ideal incompressible fluid, d'Alembert's paradox is explained by the neglect of the viscosity. Helmholtz noticed in that d'Alembert's paradox may be avoided (even if ones assumes that the fluid is ideal) if we consider that a wake (a "stagnation zone" where the velocity vanishes and the pressure is constant) appears behind the obstacle. At the beginning of the XX-th century, Levi-Cività [16] and Villat [20] developed the mathematical fundamentals of the wake flow. In 1934 Leray [14] used functional methods (topological degree theory) in order to investigate the free-boundary flow past a class of curvilinear obstacles. Numerical methods were employed for studying the wake flow model (see for example Hureau et all. [9–11]). Some of the numerical methods rely on the numerical solutions of the integral equations obtained firstly by Villat [20], and then by many other researchers. A special attention was paid to the Hammerstein type integral equation obtained for the case of Helmholtz free boundary flow past an obstacle consisting of an arc of circle, convex with respect to the direction of the uniform incoming stream. Qualitative (existence and uniqueness of the solution) and numerical results were obtained in the first half of the past century. That was not the case of the arc of circle concave with respect to the direction of the uniform at infinity stream. The integral operator has not any longer the same properties like in the case of the convex arc of circle and the numerical methods already employed in the convex case do not work for the concave one. However recent achievements in numerical analysis, namely the *arclength continuation method*, enable us to successfully investigate the Helmholtz flow past the concave arc of circle. Tracking the solution path for negative values of λ revealed a turning point on the bifurcation diagram at $\lambda_T \approx -0.218873$, below which value the Hammerstein equation has no solution. The two solutions for $\lambda_T < \lambda < 0$ correspond

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to two concave obstacles with different lengths. This method also confirmed the already theoretically known results (for positive values of λ) that there is no other real bifurcating solution out of the branch we have drawn.

Thus we are able to study an important technical application of the free-boundary fluid flow, the *vertical axis wind turbine*. One type of such wind turbine (Savonius turbine) consists of pairs of blades, symmetrical with respect to the vertical axis, the cross-section of a blade representing an arc of circle. When one blade of the pair is convex with respect to the direction of the blowing wind, the other one is concave. The aerodynamic force exerted onto the concave blade surpasses the aerodynamic force exerted onto the convex one and a torque appears. The fluid motion is unsteady (because of the rotation of the blades), but experience has shown that the shape of the blades which ensures the maximum torque in the stationary case also ensures the maximum torque for the non-stationary case. The distance between the blades is assumed to be large enough, such that there is no interaction between their wakes. The aim of the paper is to find the measure of the arc of circle representing the cross section of the blades, for which the torque is maximum in the stationary case. In fact, we prove that the value of 110° is the measure corresponding to the maximum torque. As is known [5,19], the 110° value of the measure of the arc of circle has also another significance: in case that the obstacle is a convex arc of circle whose measure surpasses 110°, the free lines detach not at the end points of the obstacle, but in two points located on the obstacle such that the measure of the arc comprised between the two detachment points is just 110°. In the detachment points the curvature of the obstacle and the curvature of the free line coincide (Brillouin's condition).

We restrict our investigations to the case of incompressible, non-viscous, stationary flow. Future attempts should take into consideration the effects of compressibility and viscosity and should treat the motion of the fluid as a non-stationary one. We mention the papers of Riabouchinsky [18] and Jacob [12] who took into account the compressibility effects for the free-boundary flow and the papers of Bassanini [4] and Hureau and Legallais [10] who combined the boundary layer theory and the free-streamline flow theory in order to obtain drag coefficients close to the coefficients obtained from numerical experiments.

The paper is organized as follows. The Helmholtz model for the free boundary flow with a wake behind the obstacle is presented in Section 2. Levi-Cività's method based on conformal mappings of the flow domain onto canonical domains expressed by means of Levi-Cività's function ω is then presented in Section 3. The conformal mapping problem is reduced by means of some integral representations to the study of a system of nonlinear integral equations in Sections 4–6. Section 7 is dedicated to the obstacles consisting of arcs of circles. The system of integral equations is reduced to a Hammerstein equation depending on a real parameter λ. The obstacle is convex with respect to the incoming fluid for $\lambda > 0$ and concave for $\lambda < 0$. One recalls some known existence and uniqueness results for positive values of λ. For negative values of this parameter, the investigation of the integral equation becomes a difficult task. We therefore employ the arclength continuation numerical method in order to solve this issue. After building the discretized algebraic system depending on λ in the beginning of Section 8, we then present in a more general context the concept of implicitly defined curves associated to such nonlinear system. The arclength continuation based on predictor–corrector methods is then described in detail in subSection 8.3 in order to numerically trace the solution curve of the Hammerstein equation. The numerical results including the bifurcation diagram, the turning point λ_T, the flow past the concave and convex arcs of o circle, as well as the two distinct solutions for each $\lambda_T < \lambda < 0$ are presented in subSection 8.4 In Section 9, we compare the drag coefficients for concave and convex arcs of circle with the same measure and conclude that 110° (i.e., 1.91986 radians) is the optimal measure for the blade section of a vertical axis drag-type wind turbine.

2. Helmholtz's model

According to this model, behind the fixed obstacle there is a wake where the fluid is at rest and the pressure is constant. Applying Bernoulli's law, we have

$$p + \frac{1}{2}\rho V^2 = p_\infty + \frac{1}{2}\rho V_\infty^2 = p_{\max}, \tag{1}$$

where p is the pressure, ρ is the density, V is the modulus of velocity, p_∞ is the pressure at infinity upstream, V_∞ is the modulus of the velocity at infinity upstream, and $p_{\max}$ is the maximum value of the pressure. In Fig. 1, we present an obstacle with the wake and the two free lines λ_1 and λ_2. Q is an isolated stagnation point, and A and B are the detachment points.

Helmholtz's model leads to a free-boundary value problem for a domain $\mathcal{D}$ from the Oxy-plane. The unknowns are the coordinates of the velocity $u, v : \mathcal{D} \to \mathbb{R}$ which are assumed to be differentiable, with continuous derivatives. We also use the complex potential φ and the stream function ψ which are connected through the relations

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}, \tag{2}$$

arising from the condition of irrotationality and from the equation of continuity (mass conservation):

$$rot\mathbf{v} = 0, \quad div\mathbf{v} = 0, \quad \mathbf{v} = (u, v). \tag{3}$$

One can easily observe that the functions φ and ψ are harmonic in $\mathcal{D}$. The obstacle $\varpi = \varpi_1 \cup \varpi_2$ and the free lines λ_1 and λ_2 are stream-lines i.e.