Applied Mathematics and Computation 230 (2014) 218–230

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Series solutions to linear integral equations

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ARTICLE INFO

Keywords:
Differential equations
Indicial equations
Integral equations

ABSTRACT

Linear Volterra-type integral equations with kernels having a series expansion in the first variable have series solutions with coefficients given iteratively. Their resolvents may be expanded likewise. The associated homogeneous equation $\mathscr{K}\mathbf{f} = \mathbf{f}$ generally has Frobenius series solutions when the kernel is singular, whereas $\mathscr{K}\mathbf{f} = \mathbf{0}$ generally has such solutions regardless of singularity: the proviso in each case is that associated "indicial equation" has solutions.

1. Introduction and summary

Volterra integral equations are a special type of integral equations introduced by Vito Volterra. They have applications in demography, the study of viscoelastic materials, and in insurance mathematics through the renewal equation. There has been a great deal of developments on the theory and applications of Volterra integral equations. For most excellent and comprehensive accounts, we refer the readers to [1–5,7]. One of the most recent developments on Volterra integral equations is described in [9].

In this paper, we show how to obtain solutions, $\mathbf{f}$, to the linear integral equation

$$\epsilon\mathbf{f} - \mathscr{K}\mathbf{f} = \mathbf{g} \tag{1.1}$$

on $\Omega \subset \mathbb{C}^p$, where ϵ lies in $\mathbb{C}$, $\mathbf{f}$ and $\mathbf{g} : \Omega \to \mathbb{C}^q$,

$$\mathscr{K}\mathbf{f}(\mathbf{x}) = \int_{\mathbf{x}T} \mathbf{K}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})d\mathbf{y},$$

where $T \subset \mathbb{C}^p$ satisfies $\Omega T \subset \Omega$ and $\mathbf{K} : \Omega \times \Omega \to \mathbb{C}^{q\times q}$ can be expanded as

$$\mathbf{K}(\mathbf{x},\mathbf{xt}) = \sum_{\mathbf{n}=\mathbf{I}-\mathbf{1}}^{\infty} \mathbf{x}^{\mathbf{n}}\mathbf{k}_{\mathbf{n}}(\mathbf{t}) \tag{1.2}$$

for $\mathbf{x}$ in Ω, $\mathbf{t}$ in T, $\mathbf{x}T = \{\mathbf{xt} : \mathbf{t} \text{ in } T\}$, $\Omega T = \{\mathbf{xt} : \mathbf{x} \text{ in } \Omega, \mathbf{t} \text{ in } T\}$, $\mathbf{xt} = (x_1t_1, \ldots, x_pt_p)\prime$, $\mathbf{x}^{\mathbf{y}} = x_1^{y_1}\cdots x_p^{y_p}$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^p$ with $0^0 = 1$. Also $\mathbf{x} \geqslant \mathbf{y}$ in $\mathbb{R}^p$ means $x_i \geqslant y_i$ for $1 \leqslant i \leqslant p$, $\mathbf{x} \ngeqslant \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ is false, $\mathbf{I}-\mathbf{1}$ is an integer in $\mathbb{R}^p$ and summation in (1.2) is over integers in $\mathbb{R}^p$ with $\mathbf{n} \geqslant \mathbf{I}-\mathbf{1}$.

Note that for a Volterra kernel $T = (\mathbf{0},\mathbf{1})$. For a Fredholm kernel on $\Omega = (\mathbf{0},\infty)$ or $\mathbb{R}^p$, $T = \Omega$. The major assumption is that for $\mathbf{n} \geqslant \mathbf{I}-\mathbf{1}$

$$\overline{\mathbf{k}}_{\mathbf{n}}(\mathbf{u}) = \int_T \mathbf{k}_{\mathbf{n}}(\mathbf{t})\mathbf{t}^{\mathbf{u}}d\mathbf{t} \text{ exists and is finite} \tag{1.3}$$

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.12.160

for certain $\mathbf{u}$ in $\mathbb{C}^p$ determined by $\mathbf{g}$. This rules out analytic Fredholm kernels on $\mathbb{C}^p$ or $(\mathbf{0}, \infty)$ but not analytic Volterra kernels. Since

$$\int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1} - \mathbf{t})^{-\mathbf{1}} \mathbf{t}^{\mathbf{u}} d\mathbf{t} = \infty,$$

it also rules out the Abel kernel $(\mathbf{x} - \mathbf{y})^{-\mathbf{1}}$.

The results of the paper are organized as follows. In Section 2, it is shown that subject to certain conditions (1.1) has a unique series solution with the coefficients being given by a simple iterative formula.

Section 3 gives a power series for the resolvent of $\mathbf{K}$ – and so provides an alternative solution to (1.1) when $\epsilon \neq 0$.

Section 4 gives solutions of Frobenius type $\mathbf{x}^{\mathbf{u}} \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \mathbf{x}^{\mathbf{n}} \mathbf{f}_{\mathbf{n}}$ to the homogeneous equation $\epsilon \mathbf{f} - \mathscr{K}\mathbf{f} = \mathbf{0}$ when $\epsilon = 0$ or $\mathbf{I} \leqslant \mathbf{0} : \mathbf{u}$ may be any solution in $\mathbb{C}^p$ of

$$\det\left(\epsilon \mathbf{I}_q - \overline{\mathbf{k}}_{-\mathbf{1}}(\mathbf{u})\right) = 0 \quad \text{if} \quad \mathbf{I} = \mathbf{0}, \tag{1.4}$$

where $\mathbf{I}_q$ is the $q \times q$ identity matrix, or of

$$\det \overline{\mathbf{k}}_{\mathbf{I}-\mathbf{1}}(\mathbf{u}) = 0 \tag{1.5}$$

if $\epsilon = 0$ or $\mathbf{I} \leqslant \mathbf{0} \neq \mathbf{I}$. It is interesting to note that $\mathbf{f} = \mathscr{K}\mathbf{f}$ has such solutions if and only if $\mathbf{I} \leqslant \mathbf{0}$. So, it is the singularity of $\mathbf{K}(\mathbf{x}, \mathbf{xt})$ not of $\mathbf{K}(\mathbf{x}, \mathbf{y})$ that allows for a solution. Further $\mathbf{K}(\mathbf{x}, \mathbf{xt})$ must be singular with respect to *every* $\mathbf{x}$ variable. By contrast $\mathscr{K}\mathbf{f} = \mathbf{0}$ will always have a solution if (1.5) can be satisfied. For example, for the Volterra kernel $\lambda x^{-\theta}(x - y)^{\theta - 1}$ with $\lambda > 0$, $\theta > 0$ and $p = q = 1$, $f = \mathscr{K}f$ has exactly one such solution while $\mathscr{K}f = 0$ has none.

Section 5 gives a class of kernel transformable to type (1.2), for example, the Volterra kernel $(\mathbf{x} - \mathbf{y})^{\boldsymbol{\theta}}$ with $\boldsymbol{\theta} > -\mathbf{1}$. Section 6 indicates extensions to other types of kernels. Section 7 discusses some non-trivial examples of the results in Sections 2 to 6.

The following notation will be used throughout the paper:

$$\overline{\mathbf{k}}(\mathbf{u}) = \int_T \mathbf{k}(\mathbf{t}) \mathbf{t}^{\mathbf{u}} d\mathbf{t} \text{ for } \mathbf{u} \text{ in } \mathbb{C}^p \text{ and } \mathbf{k} \text{ with domain } T,$$

$$\mathbf{n}! = n_1! \cdots n_p! \text{ for } \mathbf{n} \text{ in } \mathbb{N}^p,$$

$$(\mathbf{x} + \mathbf{1})! = \Gamma(\mathbf{x}) = \Gamma(x_1) \cdots \Gamma(x_p) \text{ for } \mathbf{x} \text{ in } \mathbb{C}^p,$$

$$B(\mathbf{x}, \mathbf{y}) = \Gamma(\mathbf{x})\Gamma(\mathbf{y})/\Gamma(\mathbf{x} + \mathbf{y}) = B(x_1, y_1) \cdots B(x_p, y_p) \text{ for } \mathbf{x}, \mathbf{y} \text{ in } \mathbb{C}^p,$$

$$\log \mathbf{x} = (\log x_1, \ldots, \log x_p)' \text{ for } \mathbf{x} \text{ in } \mathbb{C}^p,$$

$$I(A) = 1 \text{ if } A \text{ is true and } 0 \text{ if } A \text{ is false},$$

$$\prod_{j=1}^{p} l_j = 1 \text{ for } p = 0,$$

$$\sum_{\mathbf{I}}^{\mathbf{J}} = 0 \text{ unless } \mathbf{I} \leqslant \mathbf{J},$$

$$\sum_{\mathbf{n}=\mathbf{I}+}^{\mathbf{J}} \text{ sums over } \{\mathbf{n} : \mathbf{I} \leqslant \mathbf{n} \leqslant \mathbf{J}, \mathbf{n} \neq \mathbf{I}\},$$

$$\sum_{\mathbf{n}=\mathbf{I}}^{\mathbf{J}-} \text{ sums over } \{\mathbf{n} : \mathbf{I} \leqslant \mathbf{n} \leqslant \mathbf{J}, \mathbf{n} \neq \mathbf{J}\},$$

$$\sum_{\mathbf{n}=\mathbf{0}_+}^{\mathbf{J}} \text{ sums over } \{\mathbf{n} : \mathbf{0} \leqslant \mathbf{n} \leqslant \mathbf{J}, \mathbf{n} \neq \mathbf{0}\},$$

and

$$\mathbf{I}_+ = \max(\mathbf{0}, \mathbf{I}), \; \mathbf{I}_- = \max(\mathbf{0}, -\mathbf{I}) \text{ componentwise}.$$

So, $\mathbf{I} = \mathbf{I}_+ - \mathbf{I}_-$.