Applied Mathematics and Computation 230 (2014) 259–266

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Comparing two integral means for absolutely continuous functions whose absolute value of the derivative are convex and applications

Dah-Yan Hwang [a,*], Silvestru Sever Dragomir [b,c]

[a] Department of Information and Management, Taipei Chengshih University of Science and Technology, No. 2, Xueyuan Rd., Beitou, 112 Taipei, Taiwan
[b] Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
[c] School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

ARTICLE INFO

Keywords:
Ostrowski's inequality
Convex function
s-Convex function
Integral means
Special means

ABSTRACT

Some new estimates for the difference between the integral mean of a function and its mean over a subinterval are established and new applications for special means and probability density functions are also given.

1. Introduction

The classical Ostrowski type integral inequality [1] stipulates a bound for the difference between a function evaluated at an interior point and the average of the function over an interval. That is,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \leqslant \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = ess\sup_{t\in[a,b]}|f'(t)| < \infty$$

and $f : [a,b] \to R$ is a differentiable function on (a,b). Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

For various results and generalizations concerning Ostrowski's inequality, see [2–13] and the references therein.

In [14], Barnett et al. compared the difference of two integral means as in the following Theorem 1 in which the function has the first derivative bounded where is defined. The obtained results are also generalizations of (1.1) and have been applied to probability density functions, special means, Jeffreys divergence in Information Theory and the sampling of continuous streams in Statistics.

Theorem 1. *Let $f : [a,b] \to R$ be an absolutely continuous function with the property that $f' \in L_\infty[a,b]$. Then, for $a \leqslant x < y \leqslant b$, we have the inequality*

* Corresponding author.
E-mail addresses: dyhuang@tpcu.edu.tw (D.-Y. Hwang), sever.dragomir@vu.edu.au (S.S. Dragomir).
URL: http://rgmia.org/dragomir (S.S. Dragomir).

http://dx.doi.org/10.1016/j.amc.2013.12.058

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \leqslant \left\{\frac{1}{4} + \left[\frac{\frac{a+b}{2} - \frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x)\|f'\|_\infty \leqslant \frac{1}{2}(b-a-y+x)\|f'\|_\infty. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

The class of s-convex function in the second sense, usually denoted by K_s^2, was introduced by Hudzik and Maligranda [15]. This class is defined in the following way, $f : [0,\infty) \to R$ is said to be *s-convex function in the second sense* if

$$f(\alpha x + \beta y) \leqslant \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0,\infty), \alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. For example, the function $f : [0,1] \to [0,1]$ defined by $f(t) = t^s, s \in (0,1]$, is a s-convex function in the second sense. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [16], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense. Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense and $a, b \in [0,\infty)$ with $a < b$. If $f \in L[a,b]$, then the following inequality holds:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant is the best possible in the second inequality (1.3).

Recently, Alomari et al. [17] have established some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense.

The main purpose of this article, is to establish some new results related to the inequality (1.2) for the functions whose absolute value of the first derivatives are convex. In Section 3, the corresponding versions in the case that the power of the absolute value of the first derivative is s-convex in the second sense are obtained. Applying the obtained results, some new inequalities for special means and the probability density functions will be also given in Section 4 and Section 5, respectively.

For convenience, we denote

$$A = \frac{(y-x)(b-a-y+x)}{b-a}, \quad B = \frac{(x-a)(y-x)}{b-a},$$
$$I(a,b,x,y) = \frac{2(x-a)^2}{b-a} + \frac{6(x-a)^2(y-x)}{(b-a)(b-a-y+x)} - \frac{2(x-a)^3(y-x)}{(b-a)(b-a-y+x)^2} - \frac{3(x-a)(y-x)}{b-a} + \frac{(y-x)(b-a-y+x)}{b-a},$$
$$J(a,b,x,y) = \frac{2(x-a)^3(y-x)}{(b-a)(b-a-y+x)^2} - \frac{3(x-a)2(y-x)}{(b-a)} + \frac{2(b-a-y+x)(y-x)}{(b-a)} - \frac{2(b-y)^2}{b-a},$$

where $a \leqslant x < y \leqslant b$.

2. The function $|f'|$ is convex

Let $f : [a,b] \to R$ be an absolutely continuous function and $a \leqslant x < y \leqslant b$. Denote $K_{x,y} : [a,b] \to R$, the kernel given by

$$K_{x,y}(s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } s \in [a,x], \\ \frac{s-x}{y-x} + \frac{a-s}{b-a}, & \text{if } s \in (x,y), \\ \frac{b-s}{b-a}, & \text{if } s \in [y,b]. \end{cases}$$

The following lemma plays an important role in this article.

Lemma 1. *Let $f : [a,b] \to R$ be an absolutely continuous function and $a \leqslant x < y \leqslant b$. Then we have the identity*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du = {} & \frac{-(x-a)^2}{b-a}\int_0^1 tf'((1-t)a+tx)dt \\ & + \int_0^1 \left(\frac{(y-x)(b-a-y+x)}{b-a}t - \frac{(x-a)(y-x)}{b-a}\right)f'((1-t)x+ty)dt \\ & + \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt. \end{aligned} \tag{2.1}$$

Proof. Using the following identity given in [14],

$$\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du = \int_a^b K_{x,y}(s)f'(s)ds$$

and by suitable substitution of variables, we have the identity (2.1). □