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On the calculation of the acoustic eigenfrequencies for anisotropic materials with uncertainty

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ABSTRACT

In this work a numerical method, based on a boundary integral formulation, is used to calculate approximately the acoustic eigenfrequencies or eigenvalues in a bounded domain filled with an anisotropic material. We assume that the knowledge about the anisotropy is uncertain or incomplete, and we use fuzzy numbers to model it. Using the Zadeh's extension principle and affine transformations, we get approximate fuzzy solutions to the posed problem. Finally some numerical examples are shown.

1. Introduction

Anisotropic materials and their applications is an active research topic in engineering and applied mathematics. The knowledge regarding the parametric quantities of anisotropic materials is, in general, incomplete or vague, which makes it difficult to study its static and dynamic responses as well as their energy levels. In order to solve this difficulty, and to take into account that dynamical systems with certain degree of uncertainty are often modeled using Fuzzy Partial Differential Equations (FPDE) (see [1,3] for example) or Fuzzy Ordinary Differential Equations (FODE) (see [4,9] for example), a mathematical technique to compute approximately the acoustic eigenfrequencies in anisotropic materials with uncertainty is presented. The vagueness of the anisotropy is modeled by fuzzy numbers and the numerical method is based on a Boundary Element Method (BEM). The combination between both methodologies is made using the Zadeh's extension principle (see [12]).

The paper is organized as follows: In Section 2 some basic definitions of fuzzy numbers and the Zadeh's extension principle are shown. Section 3 presents the mathematical formulation of the deterministic problem associated to the computation of the acoustic eigenfrequencies using a BEM technique. In Section 4 mathematical model with uncertainty is described. Some numerical results are given in Section 5. Finally, in Section 6, the conclusions of this work are presented and also others applications of this technique are discussed.

2. Preliminary concepts

Let us denote by $K^c(\mathbb{R}^n)$ the set of all nonempty compact and convex subsets of $\mathbb{R}^n$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy subsets on $\mathbb{R}^n$ (functions from $\mathbb{R}^n$ to $[0,1]$). Let us define, $\forall u \in \mathcal{F}(\mathbb{R}^n)$,

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R}^n : u(x) \geqslant \alpha\}, & \alpha \in (0,1], \\ \text{supp}(u) = \overline{\{\mathbf{x} \in \mathbb{R}^n : u(\mathbf{x}) > 0\}}, & \alpha = 0, \end{cases} \tag{1}$$

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where $\overline{A}$ denotes the clousure of $A \subset \mathbb{R}^n$ and $\text{supp}(u)$ the support of u.

Also, let us define

$$\mathcal{F}^{kc}(\mathbb{R}^n) = \{u \in \mathcal{F}(\mathbb{R}^n) : \forall \alpha \in [0,1],\ [u]^\alpha \in K^c(\mathbb{R}^n)\}. \tag{2}$$

Definition 1. $u \in \mathcal{F}^{kc}(\mathbb{R})$ is called a fuzzy number.

Proposition 1. *For all* $u, v \in \mathcal{F}^{kc}(\mathbb{R}^n)$ *and* $\beta \in \mathbb{R}$ *we have the following properties:*

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\beta u]^\alpha = \beta [u]^\alpha, \quad \forall \alpha \in [0,1]. \tag{3}$$

Definition 2. For arbitrary $u, v \in \mathcal{F}^{kc}(\mathbb{R}^n)$ we define the extended Hausdorff distance D by

$$D(u,v) = \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha), \tag{4}$$

where

$$H(U,V) = max\left\{\sup_{u \in U} d(u,V), \sup_{v \in V} d(v,U)\right\} \quad \text{and} \quad d(u,V) = \inf_{v \in V} \|u - v\|. \tag{5}$$

2.1. The Zadeh's extension principle

One of the most important tools in fuzzy theory and its applications is the well-known Zadeh's extension principle (see [12]). In summary the Zadeh's extension principle says that each function $f : \mathbb{R}^n \to \mathbb{R}^n$ can be extended in another function $\widehat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ defined as follows:

$$\widehat{f}(u)(\mathbf{y}) = \begin{cases} \sup\limits_{\mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) = \mathbf{y}} u(\mathbf{x}), & \text{if } f^{-1}(\mathbf{y}) \neq \emptyset, \\ 0 & \text{if } f^{-1}(\mathbf{y}) = \emptyset \end{cases} \tag{6}$$

for each $\mathbf{y} \in \mathbb{R}^n$.

Proposition 2. *If* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function, then* $\widehat{f} : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ *is a well-defined function, and* $[\widehat{f}(u)]^\alpha = f([u]^\alpha)$ *for each* $u \in \mathcal{F}(\mathbb{R}^n)$ *and* $\alpha \in [0,1]$.

3. The mathematical model without uncertainty

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a nonempty, open, convex and bounded domain filled with an anisotropic material characterized by a matrix $\mathbf{S} \in M^{spd}_{n \times n}(\mathbb{R}) := \{\mathbf{A} \in M_{n \times n}(\mathbb{R}) : \mathbf{A}_{ij} = \mathbf{A}_{ji} \text{ and } \boldsymbol{v}^{\mathbf{T}} \cdot \mathbf{A} \cdot \boldsymbol{v} > 0\ \forall \boldsymbol{v} \in \mathbb{R}^n\}$. The boundary of the domain $\Gamma := \partial\Omega$ is supposed regular enough (piecewise smooth). The unit normal vector into the exterior of Ω, is denoted by $\mathbf{n}$.

Our purpose is, using integral representations, to solve the following eigenvalue and eigenvector problem: Find $\lambda \in \mathbb{R}$ and a non-null complex valued function $\vartheta : \Omega \to \mathbb{C}$ which is a solution of

$$\begin{cases} -\nabla \cdot \mathbf{S} \cdot \nabla \vartheta = \lambda \vartheta & \text{in} \quad \Omega, \\ \quad \vartheta = 0 & \text{on} \quad \Gamma. \end{cases} \tag{7}$$

The only non-null solutions of Eq. (7) are a countable sequence of pairs (λ_n, ϑ_n), $n \in \mathbb{N}$ that satisfy: $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \ldots \leqslant \lambda_l \to +\infty$ as $l \to +\infty$. Therefore if $\lambda \neq \lambda_n,\ \forall n \geqslant 1$, then $\vartheta(\cdot) = 0$ is the unique solution of (7).

In order to simplify the computation of problem (7), the following result is used:

Theorem 1. *The problem (7) is equivalent to problem: Find* $\lambda \in \mathbb{R}$ *and a non-null complex valued function* $\vartheta : \Omega_{\mathbf{S}} \to \mathbb{C}$ *which is a solution of*

$$\begin{cases} -\triangle \vartheta = \lambda \vartheta & \text{in} \quad \Omega_{\mathbf{S}}, \\ \quad \vartheta = 0 & \text{on} \quad \Gamma_{\mathbf{S}}, \end{cases} \tag{8}$$

where $\Omega_{\mathbf{S}}$ *is the original bounded domain modified geometrically by the affine transformation, and* $\Gamma_{\mathbf{S}} \equiv \partial\Omega_{\mathbf{S}}$ *its modified boundary respectively.*

Proof. Since $\mathbf{S} \in M^{spd}_{n \times n}(\mathbb{R})$ is a symmetric and defined positive matrix (invertible matrix), the problem (7) (see [8]) can be transformed affinely using a symmetric matrix $\mathbf{S}^{-1/2}$ as: