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Wavelet operational matrix method for solving fractional differential equations with variable coefficients

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ABSTRACT

In this paper, another operational matrix method based on Haar wavelet is proposed to solve the fractional differential equations with variable coefficients. The Haar wavelet operational matrix of fractional order integration is derived without using the block pulse functions considered in Li and Zhao (2010) [1]. The operational matrix of fractional order integration is utilized to reduce the initial equations to a system of algebraic equations. Some examples are included to demonstrate the validity and applicability of the method. Moreover, compared with the known technique, the methodology is shown to be much more efficient and accurate.

1. Introduction

Generalized from integer order equations, fractional differential ones are achieved by replacing integer order derivatives by fractional ones. Compared with integer order differential equations, they have many advantages over the capability of simulating natural physical process and dynamic system more accurately [2–4]. A large class of dynamical systems appearing throughout the field of engineering and applied mathematics can be described by fractional differential equations [5,6]. Since the increasing applications, a considerable attention has been drawn to exact and numerical solutions of differential equations with fractional order. The analytical solutions of fractional differential equations are still in a preliminary stage. However, it is difficult to obtain their exact solutions. In recent years, both physicists and mathematicians have engaged in studying the numerical methods for solving fractional differential equations. The most commonly used ones are Variational Iteration Method [7], Adomian Decomposition Method [8,9], Generalized Differential Transform Method [10,11], Homtopy Analysis Method [12], Finite Difference Method [13] and Wavelet Method [14,15].

Recently, the operational matrices of fractional order integration for the Legendre wavelet [16], Chebyshev wavelet [17], Haar wavelet [1], CAS wavelet [18] and the second kind Chebyshev wavelet have been developed to solve the fractional differential equations. All of the above mentioned wavelet methods consider the block pulse functions to obtain the operational matrices of fractional order integration.

There have been several methods for solving the fractional differential equations with variable coefficients. Bhrawy et al. [19] used quadrature tau method to solve the fractional differential equations with variable coefficients. Roberto Garra [20] applied operatorial methods to solve the analytical solution of a class of fractional differential equations with variable coefficients.

In this paper, our purpose is to proposed another method different from that in Ref. [1] based upon the Haar wavelet to solve the fractional differential equations with variable coefficients. The fractional derivative is considered in the Caputo sense [21]. We introduce the operational matrix of fractional order integration without using the block pulse functions.

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Compared with the operational matrix of fractional order integration in Ref. [1], it can improve the degree of accuracy. We adopt the orthogonal Haar wavelet matrix which is different from that in Ref. [22]. In this way, the inverse of Haar wavelet matrix do not need to calculate.

2. Definitions of fractional derivatives and integrals

In this section, we give some necessary definitions and preliminaries of the fractional calculus theory which will be used in this article [21].

Definition 1. The Riemann–Liouville fractional integral operator J^α of order α is given by

$$J^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-T)^{\alpha-1} u(T) dT, \quad \alpha > 0 \tag{1}$$

$$J^0 u(t) = u(t) \tag{2}$$

Definition 2. The Caputo definition of fractional differential operator is given by

$$D_*^\alpha u(t) = \begin{cases} \frac{d^r u(t)}{dt^r}, & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{u^{(r)}(T)}{(t-T)^{\alpha-r+1}} dT, & 0 \le r-1 < \alpha < r. \end{cases} \tag{3}$$

The Caputo fractional derivatives of order α is also defined as $D_*^\alpha u(t) = J^{r-\alpha} D^r u(t)$, where D^r is the usual integer differential operator of order r. The relation between the Riemann–Liouville integral operator J^α and Caputo differential operator D_*^α is given by the following expressions:

$$D_*^\alpha J^\alpha u(t) = u(t) \tag{4}$$

$$J^\alpha D_*^\alpha u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0 \tag{5}$$

3. Haar wavelet and function approximation

For $t \in [0, 1]$, Haar wavelet functions are defined as follows [14]

$$h_0(t) = \frac{1}{\sqrt{m}} \tag{6}$$

$$h_i(t) = \frac{1}{\sqrt{m}} \begin{cases} 2^{j/2}, & \frac{k-1}{2^j} \le t < \frac{k-(1/2)}{2^j} \\ -2^{j/2}, & \frac{k-(1/2)}{2^j} \le t < \frac{k}{2^j} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

where $i = 0, 1, 2, \ldots, m-1$, $m = 2^{p+1}$, $p = 0, 1, 2, \ldots, j$. j and k represent integer decomposition of the index i, i.e. $i = 2^j + k - 1$.

For an arbitrary function $u(t) \in L^2([0, 1))$, it can be expanded into Haar series by

$$u(t) = \sum_{i=0}^{\infty} c_i h_i(t) \tag{8}$$

where $c_i = \langle u(t), h_i(t) \rangle = \int_0^1 u(t) h_i(t) dt$ are wavelet coefficients.

In practice, only the first m terms of Eq. (8) are considered, where m is a power of 2. So we have

$$u(t) \cong \sum_{i=0}^{m-1} c_i h_i(t) \tag{9}$$

The matrix form of Eq. (9) is

$$\boldsymbol{U} = \boldsymbol{C}^T \cdot \boldsymbol{H} \tag{10}$$

where $\boldsymbol{C} = [c_0, c_1, \ldots, c_{m-1}]^T$. The row vector $\boldsymbol{U}$ is the discrete form of the function $u(t)$. $\boldsymbol{H}$ is the Haar wavelet matrix of order $m = 2^{p+1}$, $p = 0, 1, 2, \ldots, j$ i.e.