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Positive solutions of nonlinear operator equations with sign-changing kernel and its applications

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ABSTRACT

In this paper, the existence of positive solutions of nonlinear operator equation $u = \lambda Tu$ are studied, where $\lambda > 0$ is a parameter, T is a compact operator with sign-changing kernel. By using the Leray–Schauder degree theory, the existence of positive solutions of nonlinear operator equations with sign-changing kernel are obtained. The obtained abstract theorem can be applied easily, to illustrate this point the paper also introduces some applications.

1. Introduction

In this paper, we consider the existence of positive solutions of nonlinear operator equation

$$u = \lambda Tu, \tag{1.1}$$

where $\lambda > 0$ is a parameter, T has the following form:

$$(Tu)(t) = \int_{\Omega} G(t,s)g(s)f(u(s))\,ds, \quad \forall u \in X, \tag{1.2}$$

$\Omega \subset \mathbb{R}^n$ is bounded domain, and T is a compact operator on Banach space $X := C(\overline{\Omega})$, g is a continuous function, $f : [0,\infty) \to \mathbb{R}$ is continuous, $G : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ is so called integral kernel.

As we know, many boundary value problems can be written as the form of (1.1), for example (see [1–6]). Under suitable condition of nonlinear term and the condition of integral kernel $G(t,s) \geqslant 0$ on $\overline{\Omega} \times \overline{\Omega}$, Krasnoselskii's fixed point theorem (see [7,8]) can be used to prove the existence and multiplicity of positive solutions of boundary value problem, for example (see [1–4]). To be our best knowledge, however, when G change its sign, there is no general theorem to deal with the existence of positive solutions of nonlinear operator equation (1.1).

In [9], Hai obtained the existence of positive solutions to a class of elliptic boundary value problems by using the Leray–Schauder fixed point theorem. In [10], Ma studied the existence and nonexistence of positive solutions of nonlinear periodic boundary value problem with sign-changing Green's function. His approach is based on the Schauder fixed point theorem. Motivated by above references, this article is an attempt to establish an abstract theorem which can be used to prove the existence of positive solution of nonlinear operator equation (1.1) with sign-changing integral kernel. Furthermore, our aim is to classify a class of nonlinear operator equations in order to research them more extensively. At the end, the paper also introduces some applications of our obtained abstract result.

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2. Abstract theorem

In this section, we iterate and prove our main result.

We consider (1.1) under the following assumptions:

(H1) $T : X \to X$ is compact operator.
(H2) $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(0) > 0$.
(H3) $g : \overline{\Omega} \to \mathbb{R}$ is continuous, and there exists a number $k > 1$ such

$$\int_{\Omega} (G(t,s)g(s))^{+} \, ds \geqslant k \int_{\Omega} (G(t,s)g(s))^{-} \, ds, \quad t \in \overline{\Omega},$$

where

$$(G(t,s)g(s))^{+} = \begin{cases} G(t,s)g(s), & s \in \{\tilde{s} \in \overline{\Omega} \mid G(t,\tilde{s})g(\tilde{s}) \geqslant 0\}, \\ 0 & s \in \{\tilde{s} \in \overline{\Omega} \mid G(t,\tilde{s})g(\tilde{s}) < 0\}, \end{cases} \quad \text{for } t \in \overline{\Omega},$$

$$(G(t,s)g(s))^{-} = \begin{cases} -G(t,s)g(s), & s \in \{\tilde{s} \in \overline{\Omega} \mid G(t,\tilde{s})g(\tilde{s}) \leq 0\}, \\ 0, & s \in \{\tilde{s} \in \overline{\Omega} \mid G(t,\tilde{s})g(\tilde{s}) > 0\}, \end{cases} \quad \text{for } t \in \overline{\Omega}.$$

The main result in this article is given by the following abstract theorem, which provides the existence positive solution of the nonlinear operator equation (1.1) with sign-changing integral kernel.

Theorem 1.1. *Assume that* (H1), (H2) *and* (H3) *hold. Then there exists a positive number* λ^* *such that* (1.1) *has a positive solution for* $\lambda < \lambda^*$.

Proof. We shall apply Leray–Schauder degree theory to prove that (1.1) has a positive solution for $\lambda < \lambda^*$.

For $u \in X$, define the operator T_λ^+ by

$$(T_\lambda^+ u)(t) = \lambda \int_{\Omega} (G(t,s)g(s))^{+} f(u(s)) \, ds. \tag{2.1}$$

It's not difficult to see from (H1) that $T_\lambda^+ : X \to X$ is compact. Throughout this paper, we let $f(z) = 0, \ z < 0$.

Step 1. We prove that there exists a positive number $\bar{\lambda}$ such that for $0 < \lambda < \bar{\lambda}$, the equation

$$u = T_\lambda^+ u$$

has a positive solution $\tilde{u}_\lambda$ with $||\tilde{u}_\lambda||_X \to 0$ as $\lambda \to 0$, and

$$\tilde{u}_\lambda(t) \geqslant \lambda \delta f(0) p(t), \quad t \in \overline{\Omega}, \tag{2.2}$$

where δ is a positive number, $p(t) = \int_{\Omega} (G(t,s)g(s))^{+} \, ds$.

Fix $\delta \in (0,1)$, let $\varepsilon > 0$ such that

$$f(z) \geqslant \delta f(0), \quad \text{for } 0 \leqslant z \leqslant \varepsilon. \tag{2.3}$$

Suppose that

$$\lambda < \frac{\varepsilon}{2||p||_X \tilde{f}(\varepsilon)}, \tag{2.4}$$

where $\tilde{f}(z) = \max_{0 \leqslant s \leqslant z} f(s)$. Then from (2.3) and (2.4), for $0 \leqslant z \leqslant \varepsilon$

$$\frac{\tilde{f}(\varepsilon)}{\varepsilon} < \frac{1}{2\lambda ||p||_X}, \tag{2.5}$$

$$\frac{\tilde{f}(z)}{z} \geqslant \frac{\delta f(0)}{z} \to +\infty, \quad \text{as } z \to 0. \tag{2.6}$$

From (2.5) and (2.6) and (H2) there exists $A_\lambda \in (0, \varepsilon)$ such that

$$\frac{\tilde{f}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda ||p||_X}. \tag{2.7}$$