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Residue harmonic balance solution procedure to nonlinear delay differential systems

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ABSTRACT

This paper develops the residue harmonic balance solution procedure to predict the bifurcated periodic solutions of some autonomous delay differential systems at and after Hopf bifurcation. In this solution procedure, the zeroth-order solution employs just one Fourier term. The unbalanced residues due to Fourier truncation are considered by solving linear equation iteratively to improve the accuracy. The number of Fourier terms is increased automatically. The well-known sunflower equation and van der Pol equation with unit delay are given as numerical examples. Their solutions are verified for a wide range of system parameters. Comparison with those available shows that the residue harmonic balance method is effective to solve the autonomous delay differential equations. Moreover, the present method works not only in determining the amplitude but also the frequency at bifurcation.

1. Introduction

Over the past decades, the study of delay differential equations in science and engineering such as electronics [1], optics [2], biology [3], and mechanics [4] is rather extensive. A nontrivial periodic solution of a homogeneous ordinary differential equation can give rise to a Hopf bifurcation when an eigenvalue crosses the imaginary axis from left to right which corresponds to a physical parameter passing through a critical value [5]. Likewise, for delay differential equation, a general description of Hopf bifurcation can be found in and the existence can be determined from the linear stability analysis [5–8]. The new bifurcated branch can be found after the Hopf bifurcation to a steady state solution by means of the center manifold theory [6]. However, it is in general tedious to obtain good accurate periodic solutions for some nonlinear delay differential systems. As a result, many efforts have been made to find their approximate solutions [9–26,34,35]. The singular perturbation methods such as the method of averaging [9,10] multiple scales [10–12] Poincaré–Lindstedt method [12–14], Krylov–Bogoliubov–Mitropolskii method [14] have been proposed to study the delay differential equations. Compared with the center manifold reduction, the singular perturbation methods can yield accurate results for weakly nonlinear system. However, the singular perturbation methods are restricted to the small (or large) parameter assumption. In recent years, the homotopy perturbation, homotopy analysis [16–18] perturbation incremental [19,20], variational iteration [21–23] and pseudo-oscillator analysis [10,24–26] are the most widely used methods to solve the delay differential equation. Each of the methods has advantages over the others.

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When the steady state periodic solution is concerned, two categories of solution methods are available without the assumption of small nonlinearity. One is to integrate it numerically for a variety of initial conditions using one of the time-difference methods [27,28]. Very often, a large number of time steps is required to reach the steady state and different initial conditions are necessary to find different solution branches. Some solutions may be missed completely and the exact locations of the bifurcation points cannot be determined accurately due to numerical stability problems. Another category is the Galerkin technique including the harmonic balance method [29] and several of its variants [30–33].The harmonic balance technique has been applied to obtain the lower Fourier order solutions for some special cases in the earlier researches of delay differential equations [34,35]. However, it is usually difficult to achieve higher-order analytical harmonic solutions. Although, the Newton-harmonic balance [32,33] has been successful to obtain higher-order harmonic approximations to some autonomous ordinary differential equations by linearization and Newton iteration. To the best knowledge of the authors, there has not been reported on the application of Newton-harmonic balance to delay differential equation. When seeking a solution in series form, the number of terms to get a satisfactory solution is not usually known in advance. The solution process is required to repeat if the number of terms changes. Here, a hierarchical formulation is proposed so that when an approximation is available, the number of terms can be increased automatically by solving a set of linear equations. The solution of the linear equations is rather straight forwards. When applying to harmonic balance for periodic solutions, the residue harmonic balance method [36,37] has been successfully applied to ordinary differential equations and fractional order oscillator systems. In this paper, we will extend it to delay differential equation.

This paper aims at presenting a general framework of the residue harmonic balance method for calculating the periodic solutions of delay differential equations. It begins in Section 2 with a description of the residue harmonic balance method for autonomous delay differential equation and the procedure of various orders solutions. In Section 3, the numerical examples of sunflower equation and van der Pol equation with unit delay are discussed and compared. Finally, we summarize our results with a conclusion in Section 4.

2. Residue harmonic balance approximations for autonomous delay differential systems

Consider the following a class of second order autonomous delay differential equations

$$u''(t) = F(u, u', u_\tau, u'_\tau, \lambda) \tag{1}$$

where $u_\tau = u(t-\tau)$, $u'_\tau = u'(t-\tau)$, and $\tau > 0$ is the delay. A prime over a function denotes differentiation with respect to its argument. F is sufficiently differentiable and has the symmetry conditions

$$F(0,0,0,0,\lambda) = 0, \quad F(-u, -u', -u_\tau, -u'_\tau, \lambda) = -F(u, u', u_\tau, u'_\tau, \lambda) \tag{2}$$

Eq. (1) is assumed to admit a Hopf bifurcation at $\lambda = \lambda_0$. The existence of the bifurcation can be characterized by the location of the root of the characteristic function of the linear part of Eq. (1) at $u = 0$. In this paper, we investigate the bifurcated periodic solution after the Hopf bifurcation. Since the period is unknown, we normalize the period to 2π by introducing a time transformation $x = \omega t$ where the frequency ω is to be determined. Then we put Eq. (1) in the form

$$\omega^2 \ddot{u}(x) = F(u, \omega\dot{u}, u_{\tau\omega}, \omega\dot{u}_{\tau\omega}, \lambda) \tag{3}$$

where the over dot represents differentiation with respect to x. We are required to determine both the frequency and amplitude of the bifurcated periodic solution. The residue harmonic balance method is extended for purpose. The method of homotopy requires a bookkeeping parameter p with values in the interval $[0,1]$ and assumes the solution in the form

$$\begin{aligned} u(x) &= u_0(x) + pu_1(x) + p^2u_2(x) + \ldots, \\ \omega &= \omega_0 + p\omega_1 + p^2\omega_2 + \ldots \end{aligned} \tag{4}$$

where ω_i and $u_i (i = 0, 1, 2, \ldots)$ are unknowns to be determined. From condition (2), the unknown functions $u_k(x)$ can be expressed in Fourier series

$$u_k(x) = \sum_{j=0}^{k} \{a_{2j+1,k}\cos[(2j+1)x] + b_{2j+1,k}\sin[(2j+1)x]\}, \quad k = 0, 1, 2, \ldots, \tag{5}$$

and

$$\begin{aligned} u_k(x-\tau\omega) = &\sum_{j=0}^{k} \{a_{2j+1,k}\cos[(2j+1)\tau\omega] - b_{2j+1,k}\sin[(2j+1)\tau\omega\}\cos[(2j+1)x] + \sum_{j=0}^{k}\{a_{2j+1,k}\sin[(2j+1)\tau\omega] \\ &+ b_{2j+1,k}\cos[(2j+1)\tau\omega\}\sin[(2j+1)x], \quad k = 0, 1, 2, \ldots, \end{aligned}$$

where the second subscripts of coefficients a and b denote the order of corrections so that at $p = 1$, the mth-order solutions are given by

$$u^{(m)}(x) = \sum_{k=0}^{m} u_k(x), \quad \omega^{(m)} = \sum_{k=0}^{m} \omega_k. \tag{6}$$