Applied Mathematics and Computation 237 (2014) 85–97

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A mixed integer linear programming model and variable neighborhood search for Maximally Balanced Connected Partition Problem

Dragan Matić

University of Banjaluka, Faculty of Mathematics and Natural Sciences, Mladena Stojanovića 2, 78000 Banjaluka, Bosnia and Herzegovina

ARTICLE INFO

Keywords:
Balanced graphs
Graph partitioning
Mixed integer linear programming
Variable neighborhood search
Combinatorial optimization

ABSTRACT

This paper deals with Maximally Balanced Connected Partition (MBCP) problem. It introduces a mixed integer linear programming (MILP) formulation of the problem with polynomial number of variables and constraints. Also, a variable neighborhood search (VNS) technique for solving this problem is presented. The VNS implements the suitable neighborhoods based on changing the component for an increasing number of vertices. An efficient implementation of the local search procedure yields a relatively short running time. The numerical experiments are made on instances known in the literature. Based on the MILP model, tests are run using two MILP solvers, CPLEX and Gurobi. It is shown that both solvers succeed in finding optimal solutions for all smaller and most of medium scale instances. Proposed VNS reaches most of the optimal solutions. The algorithm is also successfully tested on large scale problem instances for which optimal solutions are not known.

1. Introduction

Let $G=(V,E),\ V=\{1,2,\ldots,n\}$ be a connected graph, $|E|=m$ and let w_i be weights on vertices. For any subset $V'\subset V$ the value $w(V')$ is defined as a sum of weights of all vertices belonging to V', i.e. $w(V')=\sum_{i\in V'}w_i$. The problem considered in this paper (Maximally Balanced Connected Partition Problem – MBCP) is to find a partition (V_1,V_2) of V into nonempty disjoint sets V_1 and V_2 such that subgraphs of G induced by V_1 and V_2 are connected and the value $obj(V_1,V_2)=|w(V_1)-w(V_2)|$ is minimized.

Example 1. Let us consider the graph shown in Fig. 1. The labels of the vertices are $1,2,\ldots,6$, while the weights are given in brackets, near to the labels. The total sum of weights is 30. Ideally, the partition would contain two components with the equal weights (15). In that case, the vertex 5, with the weight 10, would be put in the component with the vertex 1, or with the vertices 3 and 6, but in both cases, the components $\{1,5\}$ and $\{3,5,6\}$ are not connected. So, the optimal solution can not be equal to zero, but at least 2, because the total sum of all weights is even. One optimal solution is $(V_1,V_2)=(\{3,4,5\},\{1,2,6\})$, and the $obj(V_1,V_2)=|16-14|=2$. The other optimal solution is $(V'_1,V'_2)=(\{1,2,3,6\},\{4,5\})$, with the equal $obj(V'_1,V'_2)=|16-14|=2$.

E-mail address: matic.dragan@gmail.com

http://dx.doi.org/10.1016/j.amc.2014.03.098

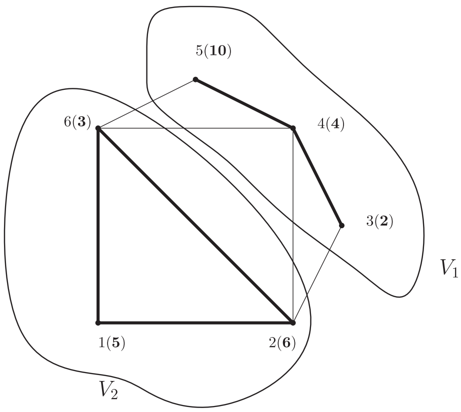

Fig. 1. The graph and the solution of the MBCP.

The first notable theoretical results in analyzing this problem are presented in [1]. In this paper, the author proved that this problem is NP hard and suggested a simple polynomial time approximation algorithm with an excellent guaranteed bound 1.072.

A successful implementation of genetic algorithm (GA) for solving this problem is presented in [2]. The authors proposed the GA using binary representation of a chromosome, indicating to which component each vertex is assigned. In the case of unconnected components (therefore incorrect solutions), penalty function is applied, directing the GA towards correct solution space. The reliability and the effectiveness of the GA is tested on grid graph and random generated instances, containing up to 300 vertices and 2000 edges.

The more general problem is so called BCPq (Balanced Connected Partition of graphs for q partitions, where $q \geqslant 2$). For a given integer number q, connected vertex weighted graph has to be partitioned into q partitions $V_1, \ldots, V_q$, so that each subgraph associated to each partition is connected and the weight of the lightest one is the highest possible, i.e., the distribution of the weights among the subgraphs should be the most homogeneous possible. In [3], 2-approximation algorithms are presented for BCPq for $q = 3$ and $q = 4$. That paper also contains a detailed proof of NP hardness of the BCPq. The authors also considered the case when q is not fixed (it is part of the instance), showing that problem does not admit an approximation algorithm with ratio smaller than 6/5, unless $P = NP$.

A similar problem to BCPq is the (l,u) partitioning, where one wishes to partition the graph G into connected components, so that the total weight of each component is at least l and at most u. Recent research in this direction is given in [4,5]. For example, in [4], the authors deal with three problems to find an (l,u)-partition of a given graph; the minimum partition problem and maximum partition problem (finding an (l,u)-partition with the minimum (respectively maximum) number of components; and the p-partition problem (finding an (l,u)-partition with a fixed number p of components). Although (l,u) partitioning is NP-hard in general case, for some special graphs it is solvable in a polynomial time. For example, in [4], the authors show that both the minimum partition problem and the maximum partition problem can be solved in time $O(u^4n)$ and the p-partition problem can be solved in time $O(p^2u^4n)$ for any series–parallel graph with n vertices. Short analysis shows that determining the existence of the (l,u) partition is a case of BCPq: If the partition for BCPq problem is found, so that the objective value of the solution is less than δ , then (l,u) partition exists for the chosen $l = t\_sum/q - \delta$ and $u = t\_sum/q + \delta$, where t_sum is the total sum of all weights.

MBCP and BCPq belong to a wide class of graph partitioning problems and have a lot of direct and closely related applications in various fields of engineering, such as digital signal processing, image processing, managing electric power networks etc., social issues and education.

For instance, for controlling and routing in large scale wireless sensor networks, the network of N clusters is considered, where each cluster corresponds to one cluster head. In order to simplify network handling, the idea is to divide that large scale network into two balanced sub-networks, which can be handled and optimized independently. The network is modelled as an undirected connected graph, $G(H,A)$, where H is the set of cluster heads, $H = \{CH_i : i = 1..N\}$ and A is the set of all undirected links (CH_i, CH_j), where CH_i and CH_j are two cluster heads. The objective is to partition G into connected balanced subgraphs and this problem corresponds to MBCP. In [6], the authors adopted the approach proposed in [1] and used it to divide the network of clusters into two smaller, connected sub-networks.

In image processing, partitioning can be used in facing image degradation, when a picture is converted from one form to another. In the situation when there is no information about the degradation process, the only way for image enhancement is to increase contrast and reduce noise by suitable modifications of the grey level of pixels. To form a graph corresponding to the image, the set of vertices corresponding to grey levels is defined, and vertex weights are defined as the number of occurrences of the corresponding tone in the image. Finding the optimal grey scale transformation is formulated as the problem of partitioning the corresponding graph, such that the sum of the weights of the vertices in each component is "as constant as possible" [7].