Applied Mathematics and Computation 237 (2014) 110–119

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Iterative penalty methods for the steady Navier–Stokes equations ☆

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ARTICLE INFO

Keywords:
Navier–Stokes equations
Iterative penalty methods
Error estimates
Inf–sup condition

ABSTRACT

This paper presents one-level and two-level iterative penalty finite element methods to approximate the solutions of steady Navier–Stokes equations. First, one-level iterative penalty finite element method is applied to solve the steady Navier–Stokes equations numerically, and its H^1 and L^2 error estimates are derived. Then, two-level iterative penalty scheme is given and its error estimates are obtained for velocity and pressure. Finally, the numerical results are displayed to verify the theoretical analysis.

1. Introduction

Let Ω be a bounded, convex, and open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary. We study the steady incompressible Navier–Stokes problem

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } \Omega,\\ \operatorname{div} u = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

which describes a steady the incompressible viscous Newtonian fluid in a bounded domain. Here, $u:\Omega\to\mathbb{R}^2$ and $p:\Omega\to\mathbb{R}$ represent the velocity and the pressure, f is the prescribed body force vector, and $\nu > 0$ is the viscosity.

Note that the velocity u and the pressure p in (1) are coupled together by the incompressibility constraint $\operatorname{div} u = 0$, which makes the system difficult to solve numerically. A popular strategy to overcome this difficulty is to relax the incompressibility constraint in an appropriate way, resulting in a class of pseudocompressibility methods, among which are the penalty method, the pressure stabilization method, the projection method, and the artificial compressibility method. In this article, we mainly consider the penalty method (see Refs. [1–14] and the references therein). This method applied to (1) is to approximate the solution (u,p) by $(u_\varepsilon, p_\varepsilon)$ satisfying the following stationary penalty Navier–Stokes equations

$$\begin{cases} -\nu\Delta u_\varepsilon + B(u_\varepsilon, u_\varepsilon) + \nabla p_\varepsilon = f, & \text{in } \Omega,\\ \operatorname{div} u_\varepsilon + \frac{\varepsilon}{\nu} p_\varepsilon = 0, & \text{in } \Omega,\\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

☆ The project is supported by NSF of China (11371289) and China Scholarship Council (201206285005).
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http://dx.doi.org/10.1016/j.amc.2014.03.112
0096-3003/

where $0 < \varepsilon < 1$ is a penalty parameter and $B(u,v) = (u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v$ is the modified bilinear term, introduced by Temam [15,16] to ensure the dissipativity of the Eq. (2). Note that in (2) can be eliminated to obtain a penalty system of only, which is much easier to solve than the original Eq. (1). Hence, the penalty method is a way to solve the Navier–Stokes problem by the single equation only contains u or p to numerically solve the original equations straightforwardly and efficiently, and has been widely used in many areas of computational fluid dynamics [2]. The other main difficulties is the nonlinear term $(u \cdot \nabla)u$, which can be precessed by the linearization method such as Newton iteration method, Stokes iteration method, Oseen iteration method [17], or the two-level methods [18–27]. Recently, the iterative penalty method was first introduced by Cheng [28] for the Stokes equations and further used to solve the pure Neumann problem [29].

In this paper, we study the one-level iterative penalty finite element method and error estimates are derived

$$\|u - u^k_{\varepsilon h}\|_1 + \|p - p^k_{\varepsilon h}\| \leqslant c(h + \varepsilon^{k+1}),$$

$$\|u - u^k_{\varepsilon h}\| \leqslant c(h^2 + \varepsilon h + \varepsilon^{k+1}).$$

Then we combine the iterative penalty method with the two-level method to approximate the solution of the problem (1). The two-level iterative penalty methods studied in this paper can be described as follows. The first step and the second step are required to solve a small Navier–Stokes equations on the coarse mesh in terms of the iterative penalty method. The third step is required to solve a large linearization problem on the fine mesh in terms of Stokes iteration. We prove that these two-level iterative penalty finite element solutions $(u_{\varepsilon h}, p_{\varepsilon h})$ are of the following error estimate

$$\|u - u_{\varepsilon h}\|_1 + \|p - p_{\varepsilon h}\| \leqslant c(h + H^2 + \varepsilon H + \varepsilon^{k+1}).$$

This paper is organized as follows. In next section, we will give the variational formulation of the problem (1) and (2) and give some notations. In Section 3, we will give the iterative penalty finite element approximation and show error estimates. In Section 4, we will propose two-level iterative penalty scheme and error estimates are proved. The numerical results is given in last section.

Throughout this paper, the symbol c always denotes some positive constant which is independent of the mesh parameter h, H, ε and that maybe depends on ν, Ω, k and the norms of u, p, f.

2. Preliminaries

Introduce

$$V = H^1_0(\Omega)^2, \quad V_\sigma = \{u \in V,\ \nabla \cdot u = 0\},$$

$$M = L^2_0(\Omega) = \left\{ q \in L^2(\Omega),\ \int_\Omega q dx = 0 \right\}.$$

Let $\|\cdot\|_k$ be the norm in Hilbert space $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and the norm in $L^2(\Omega)^2$. Then we can equip the inner product and the norm in V by $(\nabla\cdot, \nabla\cdot)$ and $\|\cdot\|_V = \|\nabla\cdot\|$, respectively, because $\|\nabla\cdot\|$ is equivalent to $\|\cdot\|_1$. Let $\mathbb{X}$ be the Banach space, denote $\mathbb{X}'$ the dual space of $\mathbb{X}$ and $<\cdot,\cdot>$ be the dual pairing in $\mathbb{X} \times \mathbb{X}'$. Introduce the following bilinear forms and trilinear form

$$\begin{cases} a(u,v) = \nu(\nabla u, \nabla v), & \forall\ u, v \in V, \\ d(v,p) = (p, \nabla \cdot v), & \forall\ v \in V, p \in M, \\ b(u,v,w) = \frac{1}{2}\int_\Omega [(u\cdot\nabla)v \cdot w - (u \cdot \nabla)w \cdot v]dx, & \forall\ u, v, w \in V. \end{cases}$$

Moreover, trilinear form $b(\cdot,\cdot,\cdot)$ satisfies

$$b(u,v,w) = -b(u,w,v), \quad \forall\ u, v, w \in V.$$

Denote

$$N = \sup_{u,v,w \in V} \frac{b(u,v,w)}{\|u\|_V \|v\|_V \|w\|_V},$$

then

$$\begin{cases} b(u,v,v) = 0, & \forall\ u, v \in V, \\ b(u,v,w) \leqslant N\|u\|_V\|v\|_V\|w\|_V, & \forall\ u\, v w \in V, \\ |b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leqslant N\|u\|_V\|v\|_2\|w\|, & \forall\ u \in V,\ \ v \in H^2(\Omega)^2,\ \ w \in L^2(\Omega)^2. \end{cases} \tag{3}$$

The weak formulation associated with the problem (1) is the following variational problem: